# Increasing, continuous operations in fuzzy $\max -*$ equations and inequalities


Zofia Matusiewicz[1]
Józef Drewniak[2]

[1]University of Information Technology and Management in Rzeszów
zmatusiewicz@wsiz.rzeszow.pl
[2]Institute of Mathematics, University of Rzeszów in Rzeszów
jdrewnia@univ.rzeszow.pl



FSTA 2012

# The plan of presentation

- Introduction
  - Notation
  - Induced implications
- Systems of equations and inequalities
  - The greatest solution
  - Convexity properties
  - Reduced matrix
  - Minimal solutions
  - Family of solution
  - Example
- Algorithms of computing minimal solutions
- Example
- Bibliography

# Notations

Let $* : [0,1]^2 \to [0,1]$ and $A \in [0,1]^{m\times n}$, $b \in [0,1]^m$, $a, c \in [0,1]$. Vectors $x, y \in [0,1]^n$ are ordered by

$$(x \leqslant y) \Leftrightarrow (\underset{1\leqslant j\leqslant m}{\forall} x_j \leqslant y_j).$$

We use notation

- $a \vee b = \max(a,b),\ a \wedge b = \min(a,b),\ a,b \in [0,1]$,
- $\bigvee\limits_{1\leqslant i\leqslant n} x_i = \max\limits_{1\leqslant i\leqslant n} x_i,\ \bigwedge_{i=1}^n x_i = \min\limits_{1\leqslant i\leqslant n} x_i,\ x_i \in [0,1]$,
- $\max -*$ product of a matrix $A$ and a vector $x$ (Zadeh 1971) we call $A \circ x \in [0,1]^m$, where

$$(A \circ x)_i = \bigvee_{j=1}^{n} (a_{ij} * x_j), \quad i \in \{1, \ldots, m\}.$$

Families of solutions:

- $S_{\leqslant}(A,b,*) = \{x \in [0,1]^n : A \circ x \leqslant b\}$,
- $S_{\geqslant}(A,b,*) = \{x \in [0,1]^n : A \circ x \geqslant b\}$,
- $S(A,b,*) = \{x \in [0,1]^n : A \circ x = b\} = S_{\geqslant}(A,b,*) \cap S_{\leqslant}(A,b,*)$,
- induced implication (Drewniak 1984) $a \overset{*}{\to} c = \max\{t \in [0,1] : a * t \leqslant c\}$,
- dual induced implication $a \overset{*}{\leftarrow} c = \min\{t \in [0,1] : a * t \geqslant c\}$.

# Induced implications

**Lemma 1**

*If an increasing operation $*$ is left continuous and $1 * 0 = 0$, then it induces implication in $[0,1]$.*

**Lemma 2**

*Let $a, b \in [0,1]$, $\{t \in [0,1] : a * t \geqslant b\} \neq \emptyset$. If an increasing operation $*$ is right continuous, then exists $a \overset{*}{\leftarrow} b$.*

**Example 1**

The binary operations and theirs implications:

$$T_P(x,y) = x \cdot y, \quad a \overset{T_P}{\rightarrow} b = \begin{cases} 1, & a \leqslant b \\ \frac{b}{a}, & a > b \end{cases}$$

and $a \overset{T_P}{\leftarrow} b = \begin{cases} \frac{b}{a}, & a \neq 0 \\ 0, & a = 0 \end{cases}$ for $a \geqslant b$.

$$T_L(x,y) = 0 \vee (x + y - 1), \quad a \overset{T_L}{\rightarrow} b = 1 \wedge (1 - a + b)$$

and $a \overset{T_L}{\leftarrow} b = \begin{cases} 1 \wedge (1 - a + b), & b \neq 0 \\ 0, & b = 0 \end{cases}$, $a \geqslant b$,

$$T_M(x,y) = x \wedge y, \quad a \overset{T_M}{\rightarrow} b = \begin{cases} 1, & a \leqslant b \\ b, & a > b \end{cases}, \quad a \overset{T_M}{\leftarrow} b = b, \ a \geqslant b$$

for all $x, y, a, b \in [0,1]$.

# Convexity properties

### Lemma 3 (cf. Drewniak 1989)

*Let* $*$ *be increasing operation. Families of solutions of* $A \circ x = b$, $A \circ x \leqslant b$ *and* $A \circ x \geqslant b$ *have the convexity property, i.e.*

$$x \in S_{\leqslant}(A, b, *) \Rightarrow [\mathbf{0}, x] \subset S_{\leqslant}(A, b, *),$$
$$x \in S_{\geqslant}(A, b, *) \Rightarrow [x, \mathbf{1}] \subset S_{\geqslant}(A, b, *),$$
$$x \leqslant y,\ x, y \in S(A, b, *) \Rightarrow [x, y] \subset S(A, b, *),$$

*where* $[\mathbf{0}, x]$, $[x, y]$, $[x, \mathbf{1}]$ *are intervals in* $([0, 1]^n, \leqslant)$.

### Corollary 1

*If* $*$ *is an increasing operation, then*

- $\mathbf{1} \in S_{\geqslant}(A, b, *) \Leftrightarrow S_{\geqslant}(A, b, *) \neq \emptyset$,
- $\mathbf{0} \in S_{\leqslant}(A, b, *) \Leftrightarrow S_{\leqslant}(A, b, *) \neq \emptyset$.

### Definition 1

By greatest solutions of system $A \circ x \leqslant b$ (and $A \circ x = b$) with $\max - *$ product we call minimal elements in $S_{\geqslant}(A, b, *)$ (in $S(A, b, *)$).

### Theorem 1

*If an operation* $*$ *is increasing, left-continuous on the second argument and* $1 * 0 = 0$, *then* $S_{\leqslant}(A, b, *)$ *the complete lattice. Moreover, if* $S_{\geqslant}(A, b, *) \neq \emptyset$ *and* $S(A, b, *) \neq \emptyset$, *then* $S_{\geqslant}(A, b, *) \neq \emptyset$ *and* $S(A, b, *) \neq \emptyset$ *are closed under arbitrary suprema.*

# The greatest solution

Let $u = \max S_{\leqslant}(A, b, *) = \max\{x \in [0,1]^n : A \circ x \leqslant b\}$.

**Theorem 2**

*If an operation $*$ is increasing, left-continuous on the second argument and $1 * 0 = 0$, then there $u$ is the greatest element of $S_{\leqslant}(A, b, *)$, where*

$$u_j = \bigwedge_{i=1}^{m} (a_{ij} \xrightarrow{*} b_i), \; j \in \{1, \ldots, n\}.$$

*It means $u = A \xrightarrow{\circ} b$.*

**Corollary 2**

*If an operation $*$ is increasing, left-continuous on the second argument and $1 * 0 = 0$, then $S_{\leqslant}(A, b, *) = [\mathbf{0}, A \xrightarrow{\circ} b]$.*

**Theorem 3**

*If an operation $*$ is increasing, left-continuous on the second argument, $1 * 0 = 0$ and $S(A, b, *) \neq \emptyset$, then $\max S(A, b, *) = A \xrightarrow{\circ} b$.*

# Reduced matrix

**Definition 2**

Let $x \in S(A, b, *)$. By reduced matrix of equation system $A \circ x = b$ we call the matrix $A'_b(x)$, where

$$a'_{ij}(x) = \begin{cases} a_{ij} & \text{, if } a_{ij} * x_j = b_i \\ 0 & \text{, in other case} \end{cases}, i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}.$$

Let $x \in S_{\geqslant}(A, b, *)$. By reduced matrix of system of inequalities $A \circ x \geqslant b$ we call $A'_{\geqslant b}(x)$, where

$$a^{\geqslant}_{ij}(x) = \begin{cases} a_{ij} & \text{, if } a_{ij} * x_j \geqslant b_i \\ 0 & \text{, in other case} \end{cases}, i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}.$$

**Theorem 4 (Drewniak, Matusiewicz 2010)**

*If an operation $*$ is increasing, left-continuous on the second argument, and it has neutral element $e = 1$, then $S(A, b, *) = S(A'_b(u), b, *)$.*

# Minimal solutions (1)

**Definition 3**

By minimal solutions of system $A \circ x \geqslant b$ (and $A \circ x = b$) with $\max -*$ product we call minimal elements in $S_{\geqslant}(A, b, *)$ (in $S(A, b, *)$). The set of all minimal solution is denoted by $S^0_{\geqslant}(A, b, *)$ ($S^0(A, b, *)$).

**Corollary 3 (cf. Drewniak 1989)**

*If $*$ is increasing operation, then*

$$\bigcup_{x \in S^0_{\geqslant}(A,b,*)} [x, \mathbf{1}] \subset S_{\geqslant}(A, b, *).$$

**Theorem 5**

*If $*$ is increasing, right-continuous on the second argument, then*

- *each $x \in S_{\geqslant}(A, b, *)$ is bounded from below by some $v \in S^0_{\geqslant}(A, b, *)$,*
- *each $x \in S(A, b, *)$ is bounded from below by some $v \in S^0(A, b, *)$,*
- *we have $S^0(A, b, *) \subset S^0_{\geqslant}(A, b, *)$.*

**Theorem 6**

*If $*$ is increasing, continuous on the second argument and $1 * 0 = 0$, then*

$$S(A, b, *) = \bigcup_{v \in S^0(A,b,*)} [v, A \overset{\circ}{\to} b].$$

# Algorithm of computing minimal solutions (1)

Let $S_{\geqslant}(A, b, *) \neq \emptyset$, an operation $*$ be increasing, right-continuous on the second argument one and

$$0 < b_m \leqslant \ldots \leqslant b_2 \leqslant b_1.$$

ALGORITHM I

Step 1. Determine the reduced matrix $A'_{\geqslant b}(x)$. Let $i := 1$, $K := \emptyset$, $V := \{1, \ldots, m\}$.

Step 2. Choose $k_i$ that $a'_{ik_i} > 0$ and calculate $v_{k_i} = a'_{ik_i} \overset{*}{\leftarrow} b_i$ and $K := K \cup \{k_i\}$.

Step 3. Determine the set

$$V := V \cap \{i < s \leqslant m \text{ oraz } a'_{sk_i} * v_{k_i} < b_s\}.$$

Step 4. If $V \neq \emptyset$, to $i := \min V$ and return to Step 2.
In other case go to Step 5.

Step 5. If $k \notin K$, then $v_k := 0$.

Let us denote the set of all vectors $v$ from this algorithm obtained for $x \in S_{\geqslant}(A, b, *)$ by $Alg(x)$(see Step 2).

**Corollary 4**

*Let $x \in S_{\geqslant}(A, b, *)$. If an operation $*$ is increasing, right-continuous on the second argument, then*

$$card\, Alg(x) \leqslant m^n.$$

# Minimal solutions (2)

**Theorem 7**

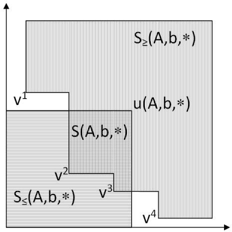
*If an operation* $*$ *is increasing, right-continuous on the second argument, then* $S^0_{\geqslant}(A, b, *) \subset Alg(\mathbf{1})$.

**Theorem 8**

*Let* $x \in S(A, b, *)$. *If an operation* $*$ *is increasing, right-continuous on the second argument, then* $Alg(x) \subset S(A, b, *)$.

**Theorem 9**

*Let* $b \in (0, 1]^n$. *If an operation* $*$ *is increasing, continuous on the second argument and* $1 * 0 = 0$, *then* $S^0(A, b, *) \subset Alg(A \overset{\circ}{\rightarrow} b)$.




$$S(A, b, *) = \bigcup_{v \in S^0(A,b,*)} [v, A \overset{\circ}{\rightarrow} b].$$

# Example 2

Let $x * y = \sqrt{x \cdot y}$ and

$A = \begin{bmatrix} 0.1 & 0.16 & 0.25 \\ 0.2 & 0.09 & 0.05 \end{bmatrix}$, $b = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}$, $A'_b(\mathbf{1}) = \begin{bmatrix} 0 & 0.16 & 0.25 \\ 0.2 & 0.09 & 0 \end{bmatrix}$.

We get $a \overset{*}{\leftarrow} b = \frac{b^2}{a}, b^2 \leqslant a$.

Using $Alg(\mathbf{1})$ we get:

1. For $k_1 = 2$ we get $K = \{2\}$ and $V = \emptyset$. We obtain $v_2^1 = 0.16 \overset{*}{\leftarrow} 0.4 = 1$.
2. For $k_1 = 3$ we get $v_3^2 = 0.2 \overset{*}{\leftarrow} 0.4 = 0.8$ and $K = \{3\}$, $V = \{2\}$, $i = 2$.
Choosing $k_2 = 1$, we compute $v_1^2 = 0.2 \overset{*}{\leftarrow} 0.3 = 0.45$, $K = \{1, 3\}$, $V = \emptyset$.
3. For $k_1 = 3$ we get $v_3^3 = 0.2 \overset{*}{\leftarrow} 0.4 = 0.8$ and $K = \{3\}$, $V = \{2\}$, $i = 2$.
Choosing $k_2 = 2$, we compute $v_2^3 = 0.09 \overset{*}{\leftarrow} 0.3 = 1$, $K = \{2, 3\}$, $V = \emptyset$.

Thus we have the following projections:

$$v^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v^2 = \begin{bmatrix} 0.45 \\ 0 \\ 0.8 \end{bmatrix}, v^3 = \begin{bmatrix} 0 \\ 1 \\ 0.8 \end{bmatrix}.$$

Since $v^1 || v^2$ and $v^1 \leqslant v^3$, then $S^0_{\geqslant}(A, b, *) = \{v^1, v^2\}$.

# Algorithm of computing minimal solutions (2)

Let an operation $*$ be increasing, continuous on the second argument and have neutral element $e = 1$.

ALGORYTM I'

*Step 0.* We calculate $u = A \xrightarrow{\circ} b$.
*Step 1.* We determine $Alg(u)$ from Algorithm I.
*Step 2.* We determine $S^0(A, b, *)$ as a set of minimal elements in $Alg(u)$.

**Definition 4**

An operation $*$ is conditionally cancellative if

$$a * x = a * y \neq 0 \Rightarrow x = y \quad \text{for } a, x, y \in (0, 1].$$

**Theorem 10**

*Let $*$ be increasing, continuous on the second argument and conditionally cancellative operation and $1 * 0 = 0$, then If $v \in S^0(A, b, *)$, then $v_j \in \{0, u_j\}$ for $j \in N$, where $u = A \xrightarrow{\circ} b$.*

**Corollary 5**

*If $*$ is increasing, continuous on the second argument and conditionally cancellative operation and $1 * 0 = 0$, then*

$$card\ S^0(A, b, *) \leqslant \binom{n}{[\frac{n}{2}]}.$$

# Example 3

Let $* = T_P$ and

$$A = \begin{bmatrix} 1 & 0.8 & 0.5 & 0.5 \\ 0.8 & 0.8 & 0.1 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.2 & 0.1 \end{bmatrix}, \quad b = \begin{bmatrix} 0.5 \\ 0.4 \\ 0.3 \\ 0.2 \end{bmatrix}.$$

We determine $Alg(u)$:

$$u = \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \\ 1 \end{bmatrix}, v^1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}, v^2 = \begin{bmatrix} 0.5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v^3 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v^4 = \begin{bmatrix} 0.5 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$v^5 = \begin{bmatrix} 0 \\ 0.5 \\ 1 \\ 0 \end{bmatrix}, v^6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, v^7 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v^8 = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 1 \end{bmatrix}, v^9 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

From Algorithm I' we obtain the solutions of $A \circ x = b$. In this set we have all minimal solution of the system. We get $S^0(A, b, *) = \{v^1, v^2, v^3, v^5, v^6, v^8\}$, because $v^2 = v^4$, $v^6 = v^9$, $v^3 = v^7$.

# Literature


- E. Czogała, J. Drewniak, W. Pedrycz, Fuzzy relation equations on a finite set, Fuzzy Sets and Systems 7 (1982), 89—101.
- E. Czogała, J. Drewniak, Associative monotonic operations in fuzzy set theory, Fuzzy Sets and Systems 12 (1984), 249—269.
- J. Drewniak, Fuzzy relation equations and inequalities, Fuzzy Sets and Systems 14 (1984), 237—247.
- J. Drewniak, Fuzzy relation calculus, Silesian University, Katowice 1989.
- J. Drewniak, Z. Matusiewicz, Properties of $\max - *$ fuzzy relation equations, Soft Computing 14 (10) (2010), 1037—1041.
- S.-C. Han, H.-X. Li, J.-Y. Wang, Resolution of finite fuzzy relation equations based on strong pseudo-t-norms, Applied Mathematics Letters 19 (8) (2006), 752-757.
- A.A. Molai, E. Khorram, An algorithm for solving fuzzy relation equations with max-T composition operator, Information Sciences 178 (2008) 1293—1308.
- B.-S. Shieh, Solutions of fuzzy relation equations based on continuous $t$-norms, Information Sciences 177(19) (2007), 4208—4215.
- G.B. Stamou, S.G. Tzafestas, Resolution of composite fuzzy relation equations based on Archimedean triangular norms, Fuzzy Sets and Systems 120 (2001) 395—407.