Algebras of Fuzzy Sets

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FSTA 2012, Liptovský Ján

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- lattice of idempotent fuzzy subsets of a groupoid
- generalization: algebras of fuzzy sets (fuzzy power algebras)-results about homomorphisms, subalgebras, direct products
- very new results: special kinds of fuzzy equalities, identities, equational classes, Birkhoff-style theorems

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- $\mathcal{F}_2(D)$ of all idempotent fuzzy subsets of a cancellative semigroup D forms a complete lattice
- results concerning the set $\mathcal{F}(D)$ of all fuzzy subsets of D and its substructures $\mathcal{F}_{-1}(D)$ and $\mathcal{F}_2(D)$.

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- $\mathcal{F}_2(D)$ of all idempotent fuzzy subsets of a cancellative semigroup D forms a complete lattice
- results concerning the set $\mathcal{F}(D)$ of all fuzzy subsets of D and its substructures $\mathcal{F}_{-1}(D)$ and $\mathcal{F}_2(D)$.
- If D is a cancellative groupoid, then the sup-min product is distributive over an arbitrary intersection of fuzzy sets in $\mathcal{F}(D)$!

Lattice of fuzzy sets of a groupoid

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$$(\lambda \cdot \mu)(x) = \sup_{x=ab} \min(\lambda(a), \mu(b)).$$

- $$\mathcal{F}_{-1}(D) = \{\lambda \in \mathcal{F}(D) \mid \lambda \subseteq \lambda \cdot \lambda\}$$
$$\mathcal{F}_2(D) = \{\lambda \in \mathcal{F}(D) \mid \lambda = \lambda \cdot \lambda\}$$

RID and LID groupoids

Definition

For a groupoid G we will say that it is *right intersection-distributive* (RID) if for any family $\{\lambda_i \mid i \in I\} \subseteq \mathcal{F}(D)$ and any $\mu \in \mathcal{F}(D)$ it holds

$$(1) \qquad (\bigcap_{i \in I} \lambda_i) \cdot \mu = \bigcap_{i \in I} (\lambda_i \cdot \mu).$$

Similarly, D is *left intersection-distributive* (LID)

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- The only RID semigroups are left-zero semigroups. The only LID semigroups are...
- A finite groupoid can not be both RID and LID.
- An infinite groupoid D is both RID and LID iff in its Cayley table every element from D appears at most once.
- There are infinite groupoids which are at the same time RID and LID.

Results

Theorem

(1) A groupoid D is RID iff D satisfies the quasiidentity

$$xy = zt \Rightarrow x = z.$$

(2) A groupoid D is LID iff D satisfies the quasiidentity

$$yx = tz \Rightarrow x = z.$$

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Theorem

Let D be a groupoid. Then $\mathcal{F}_2(D)$ is a complete lattice which is a complete join-sublattice of $\mathcal{F}_1(D)$, and a complete meet-sublattice of $\mathcal{F}_{-1}(D)$.

General problem: fuzzy power algebras

Universal algebras in fuzzy world

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- Belohlavek, Vychodil (2000, 2002, 2003, 2006)
- Vojvodic, Šešelja (1983),Tepavčević

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- *different fuzzyfication*: what is fuzzyfied - the universe, operations, equality relation,...
- We choose: universal algebras, complete residuated lattices (sometimes, complete lattices), and trying to connect different fuzzyfications...

Residuated lattices

A *residuated lattice* is an algebra $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow \rangle$ where

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- examples: on $[0, 1]$, Lukasiewicz, Gödel and product structures

Basic notions and notation

Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow\rangle$ be a complete residuated lattice, X a nonempty set.

- **set of all L-fuzzy sets on X**: $\mathcal{F}_{\mathcal{L}}(X)$, or $\mathcal{F}(X)$ or L^X

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- **fuzzy relation on** A: $\eta : A^n \rightarrow L$.
- **fuzzy equivalence**: a binary fuzzy relation on A which is
  - reflexive: $\eta(x, x) = 1$, for all $x \in A$,
  - symmetric: $\eta(x, y) = \eta(y, x)$, for $x, y \in A$,
  - transitive: $\eta(x, y) \otimes \eta(y, z) \leq \eta(x, z)$, for all $x, y, z, \in A$.

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- **fuzzy equality**: fuzzy equivalence relation η if from $\eta(x, y) = 1$ it follows $x = y$.

Extension Principle

- extension principle: how to extend a function $f : X_1 \times \cdots \times X_n \to Y$ into $\overline{f} : L^{X_1} \times \cdots \times L^{X_n} \to L^Y$ (L. Zadeh)
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- applications: fuzzy arithmetic, engineering problems, analysis of discrete dynamical systems, fuzzy fractals, fuzzy transportation problems...
- Let $\mathcal{L}$ be a complete lattice or a complete residuated lattice, $f : X_1 \times \cdots \times X_n \to Y$. Define $\overline{f} : L^{X_1} \times \cdots \times L^{X_n} \to L^Y$

$$\overline{f}(\mu_1, \ldots, \mu_n)(y) = \bigvee_{\substack{x_i \in X_i \\ f(x_1, \ldots, x_n) = y}} \mu_1(x_1) \wedge \cdots \wedge \mu_n(x_n). \qquad (1)$$

- If we have a residuated lattice, $f : X_1 \times \cdots \times X_n \to Y$, we can extend it to a function $\overline{f} : L^{X_1} \times \cdots \times L^{X_n} \to L^Y$ in an alternative way:

$$\overline{f}^{\otimes}(\mu_1, \ldots, \mu_n)(y) = \bigvee_{\substack{x_i \in X_i \\ f(x_1,\ldots,x_n)=y}} \mu_1(x_1) \otimes \cdots \otimes \mu_n(x_n). \qquad (2)$$

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Algebra of fuzzy sets 1

Definition

Let $\mathcal{L}$ be a complete lattice or a residuated lattice and $\mathcal{A} = \langle A, \{f \mid f \in \Omega\}\rangle$ be a universal algebra. If $f \in \Omega$ is an n-ary fundamental operation of $\mathcal{A}$, define $\bar{f}^{\wedge} : \mathcal{F}(A)^n \to \mathcal{F}(A)$ in the following way:

$$\bar{f}^{\wedge}(\mu_1, \dots, \mu_n)(y) = \bigvee_{\substack{x_i \in X_i \\ f(x_1,\dots,x_n)=y}} \mu_1(x_1) \wedge \cdots \wedge \mu_n(x_n).$$

The algebra $\mathcal{F}^{\wedge}(\mathcal{A}) = \langle \mathcal{F}(A), \{\bar{f}^{\wedge} \mid f \in \Omega\}\rangle$ will be called the $\wedge$-**algebra of fuzzy sets induced by** $\mathcal{A}$.

Algebra of fuzzy sets 2

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The algebra $\mathcal{F}^{\otimes}(\mathcal{A}) = \langle \mathcal{F}(A), \{\bar{f}^{\otimes} \mid f \in \Omega\}\rangle$ will be called the **$\otimes$-algebra of fuzzy sets induced by $\mathcal{A}$.**

Of course, if $\mathcal{L}$ is a complete Heyting algebra, or more specially, if $L = \{0, 1\}$, the two kinds of induced algebras of fuzzy sets coincide.

Power algebras 1

- crisp case: both kinds of induced algebras of fuzzy sets become the ordinary *power algebra* of $\mathcal{A}$ (*algebra of complexes* or *global* of $\mathcal{A}$).

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- **Definition**

  Let A be a non-empty set, $\mathcal{P}(A)$ the set of all subsets of A, and $f : A^n \to A$. We define $f^+ : \mathcal{P}(A)^n \to \mathcal{P}(A)$ in the following way:

$$f^+(X_1, \ldots, X_n) = \{f(x_1, \ldots, x_n) \mid x_1 \in X_1, \ldots, x_n \in X_n\}.$$

  If $\mathcal{A} = \langle A, \{f \mid f \in \Omega\}\rangle$ is an algebra, the **power algebra** (or **complex algebra**, or **global**) $\mathcal{P}(\mathcal{A})$ is defined as:

$$\mathcal{P}(\mathcal{A}) = \langle \mathcal{P}(A), \{f^+ \mid f \in \Omega\}\rangle.$$

Power algebras 2

- crisp power algebras are used:
  - group theory, semigroup theory
  - lattices (the set of ideals of a distributive lattice L again forms a lattice, and meets and joins in the new lattice are precisely the power operations of meets and joins in L)
  - formal language theory (the product of two languages is simply the power operation of concatenation of words)
  - non-classical logics (Jonsson, Tarski, Boolean algebras with operators...)

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Proposition

Let $\mathcal{A}$ be any universal algebra, and $\mathcal{L}$ the usual two element Boolean algebra. Then both of $\wedge$-algebra and the $\otimes$-algebra of L-fuzzy sets induced by $\mathcal{A}$ coincide with the power algebra of $\mathcal{A}$, i.e.

$$\mathcal{F}^{\wedge}(\mathcal{A}) = \mathcal{F}^{\otimes}(\mathcal{A}) = \mathcal{P}(\mathcal{A}).$$

Homomorphisms 1

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type Ω. A mapping $\alpha : A \to B$ is a **homomorphism from** $\mathcal{A}$ to $\mathcal{B}$ if for all $n \geq n$, all $f \in \Omega_n$, all $a_1, a_2, \ldots, a_n \in A$,

$$\alpha(f^{\mathcal{A}}(a_1, a_2, \ldots, a_n)) = f^{\mathcal{B}}(\alpha(a_1), \alpha(a_2), \ldots, \alpha(a_n)).$$

Proposition

Let $\mathcal{L}$ be a lattice or a residuated lattice. Then:

(a) If $\alpha : A \to B$, $\beta : B \to C$, then $\overline{(\beta \circ \alpha)} = \overline{\beta} \circ \overline{\alpha}$.

(b) If $\alpha : A \to B$ is a bijection, then $\overline{\alpha} : \mathcal{F}(A) \to \mathcal{F}(B)$ is also a bijection.

Homomorphisms 2

Let $\alpha : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Will the induced mapping $\overline{\alpha} : \mathcal{F}(A) \to \mathcal{F}(B)$ be a homomorphism from $\mathcal{F}^{\otimes}(\mathcal{A})$ to $\mathcal{F}^{\otimes}(\mathcal{B})$, and from $\mathcal{F}^{\wedge}(\mathcal{A})$ to $\mathcal{F}^{\wedge}(\mathcal{B})$? The two kinds of induced algebras of fuzzy sets do not behave in the same way!

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Theorem

Let $\mathcal{L}$ be a complete residuated lattice, $\mathcal{A}$ and $\mathcal{B}$ two algebras of type Ω. If $\alpha : \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $\overline{\alpha} : \mathcal{F}^{\otimes}(\mathcal{A}) \to \mathcal{F}^{\otimes}(\mathcal{B})$ is also a homomorphism.

BUT: there are algebras $\mathcal{A}, \mathcal{B}$, a complete lattice L, such that $\alpha : \mathcal{A} \to \mathcal{B}$ is a homomorphism, but $\overline{\alpha} : \mathcal{F}^{\wedge}(\mathcal{A}) \to \mathcal{F}^{\wedge}(\mathcal{B})$ is not a homomorphism!

Example

Example

$\cdot$	u_1	u_2	v_1	v_2
u_1	v_1	v_1	u_1	u_2
u_2	v_1	v_1	u_2	u_2
v_1	v_2	v_2	v_1	v_1
v_2	v_2	v_2	v_1	v_1

$\cdot$	a	b
a	b	a
b	b	b

Let $\mathcal{L}$ be the pentagon (with $0 < r < q < 1$, $0 < p < 1$, p not being comparable to r or q). There is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ such that $\overline{\alpha} : \mathcal{F}^{\wedge}(\mathcal{A}) \to \mathcal{F}^{\wedge}(\mathcal{B})$ is not a homomorphism.

Theorem

Let $\mathcal{L}$ be a completely distributive lattice, $\mathcal{A}$ and $\mathcal{B}$ two algebras of type Ω. If $\alpha : \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $\overline{\alpha} : \mathcal{F}^{\wedge}(\mathcal{A}) \to \mathcal{F}^{\wedge}(\mathcal{B})$ is also a homomorphism.

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Theorem

Let $\mathcal{L}$ be a complete residuated lattice, and $\mathcal{A}$ a subalgebra of algebra $\mathcal{B}$. Then $\mathcal{F}^{\otimes}(\mathcal{A})$ can be embedded into the algebra $\mathcal{F}^{\otimes}(\mathcal{B})$.

Fuzzy products 1

Definition

The **fuzzy product** of the family of fuzzy subsets $\langle \eta_i \in \mathcal{F}(A_i) \mid i \in I \rangle$ is the fuzzy subset $\prod^{\wedge} \eta_i : \prod A_i \to L$ defined in the following way: if $x \in \prod A_i$, where $x_i = x(i)$ for $i \in I$, then

$$\left(\prod^{\wedge} \langle \eta_i \mid i \in I \rangle\right)(x) = \bigwedge \{\eta_i(x_i) \mid i \in iI\}.$$

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Theorem

Let $\mathcal{L}$ be a completely distributive lattice and $\langle \mathcal{A}_i \mid i \in I \rangle$ be a family of algebras of the type Ω. The mapping
$\varphi : \prod_{i \in I} \mathcal{F}(A_i) \to \mathcal{F}(\prod_{i \in I} A_i)$ defined by $\varphi(\eta) = \prod^{\wedge} \eta(i)$ is a homomorphism from $\prod_{i \in I} \mathcal{F}^{\wedge}(\mathcal{A}_i)$ to $\mathcal{F}^{\wedge}(\prod_{i \in I} \mathcal{A}_i)$.

Fuzzy products 2

It is not hard to see that the above defined mapping $\varphi : \prod_{i \in I} \mathcal{F}(A_i) \to \mathcal{F}(\prod_{i \in I} A_i)$ is not necessarily injective.

Example

Take $L = [0,1]$, $I = \{1,2,3\}$ and $\eta_1, \eta_2, \eta_3, \eta_4 : A \to L$ such that

$\eta_1(x) = 0.1, \quad$ for all $x \in A$

$\eta_2(x) = 0.1, \quad$ for all $x \in A$

$\eta_3(x) = 0.9, \quad$ for all $x \in A$

$\eta_4(x) = 0.8, \quad$ for all $x \in A$

Then $\varphi((\eta_1, \eta_2, \eta_3)) = \varphi((\eta_1, \eta_2, \eta_4)) = \mu$, where $\mu(x) = 0.1$ for all $x \in A$.

Fuzzy products 3

Definition

$\mathcal{F}_+(A) \subseteq \mathcal{F}(A)$ is defined by
$\mathcal{F}_+(A) = \{\eta : A \to L \mid (\exists x \in A)\ \eta(x) = 1\}$.

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Theorem

Let $\langle \mathcal{A}_i \mid i \in I \rangle$ be a family of sets. Then the mapping $\varphi_+ : \prod_{i \in I} \mathcal{F}_+(A_i) \to \mathcal{F}_+(\prod_{i \in I} A_i)$ defined by $\varphi_+(\eta) = \prod_{i \in I}^{\wedge} \eta(i)$ is injective.

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Theorem

Let $\mathcal{A}$ be an algebra of type Ω. Then $\mathcal{F}_+(A)$ is a subuniverse of the algebras $\mathcal{F}^{\wedge}(\mathcal{A})$ and $\mathcal{F}^{\otimes}(\mathcal{A})$.

Fuzzy products 4

It can be proved that the two kind of algebras of fuzzy sets behave in different way in respect to direct products.

Theorem

Let $\mathcal{L}$ be a completely distributive lattice and $\langle \mathcal{A}_i \mid i \in I \rangle$ be a family of algebras of the type Ω. The mapping $\varphi : \prod_{i\in I} \mathcal{F}(A_i) \to \mathcal{F}(\prod_{i\in I} A_i)$ defined by $\varphi(\eta) = \prod^{\wedge} \eta(i)$ is a homomorphism from $\prod_{i\in I} \mathcal{F}^{\wedge}(\mathcal{A}_i)$ to $\mathcal{F}^{\wedge}(\prod_{i\in I} \mathcal{A}_i)$.

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BUT: We can construct two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ such that the mapping $\varphi : \mathcal{F}(A_1) \times \mathcal{F}(A_2) \to \mathcal{F}(A_1 \times A_2)$ defined by $\varphi(\langle \eta, \mu \rangle) = \prod^{\wedge}_{i \in \{1,2\}} \eta(i)$ is not a homomorphism from $\mathcal{F}^{\otimes}(\mathcal{A}_1) \times \mathcal{F}^{\otimes}(\mathcal{A}_2)$ to $\mathcal{F}^{\otimes}(\mathcal{A}_1 \times \mathcal{A}_2)$.

Example

Example

The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ will be groupoids given respectively by their Cayley's tables:

$\cdot$	a	b
a	b	a
b	b	b

$\cdot$	c	d
c	c	d
d	c	c

Let $\mathcal{L}$ be standard residual lattice on the real unit interval [0,1] with the product structure, i.e. $\otimes$ is the usual product of real numbers. Let us take $\eta = \langle \eta_1, \eta_2 \rangle$, $\mu = \langle \mu_1, \mu_2 \rangle$, where $\eta_1(a) = 0.6$, $\eta_2(c) = 0.5$, $\mu_1(b) = 0.7$, $\mu_2(d) = 0.8$. It can be proved that

$$(\varphi(\eta \cdot \mu))(\langle a, d \rangle) \neq (\varphi(\eta) \cdot \varphi(\mu)(\langle a, d \rangle).$$

Algebras with fuzzy equalities

Proposition

Let $\mathcal{L}$ be a complete residuated lattice, $\mathcal{A}$ a universal algebra, and $\approx$ the similarity relation defined on $\mathcal{F}(A) = L^A$ by $(\eta \approx \mu) = \wedge\{\eta(x) \leftrightarrow \mu(x) \mid x \in A\}$. Then the structures $\langle \mathcal{F}^{\wedge}(\mathcal{A}), \approx \rangle$ and $\langle \mathcal{F}^{\otimes}(\mathcal{A}), \approx \rangle$ are algebras with fuzzy equality.

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Definition (Belohlavek)

Let $\mathcal{L}$ be a complete residuated lattice, $\langle A, \Omega\rangle$ be a universal algebra, and $\approx$ an fuzzy equality on A. Then the structure $\mathcal{A} = \langle A, \Omega, \approx\rangle$ is an **$\mathcal{L}$-algebra with fuzzy equality** if each operation $f \in \Omega$ is compatible with $\approx$, i.e. for any n-ary $f \in \Omega$, for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ we have $(a_1 \approx b_1) \otimes \cdots \otimes (a_n \approx b_n) \leq f(a_1, \ldots, a_n) \approx f(b_1, \ldots, b_n)$.

Fuzzy (sub)algebras and equalities

Definition (B. Šešelja, A. Tepavčević)

Let $\mathcal{A}$ be a universal algeba of type Ω, L a complete lattice, $\mu : A \to L$ a fuzzy (sub)algebra of $\mathcal{A}$. A μ-**fuzzy equality** is any binary fuzzy relation $E : A^2 \to L$ such that:

- $E(x, y) < \mu(x) = E(x, x)$, for all different $x, y \in A$,
- $E(x, y) = E(y, x)$, for all $x, y \in A$,
- $E(x, y) \wedge E(y, z) \leq E(x, z)$, for all $x, y, z \in A$,
- $E(a_1, b_1) \wedge E(a_2, b_2) \wedge \cdots \wedge E(a_n, b_n) \leq E(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n))$, for all...

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- $E(a_1, b_1) \wedge E(a_2, b_2) \wedge \cdots \wedge E(a_n, b_n) \leq$ $E(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n))$, for all...

Let $\mathcal{A}$ be a universal algeba of type Ω, L a complete lattice. Then $\mu : A \to L$ is a *fuzzy (sub)algebra* of $\mathcal{A}$ if for all...

$$\mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n) \leq \mu(f(a_1, a_2, \ldots, a_n)).$$

Right (μ, r) relation

- algebras with fuzzy equality : variety theorem (Belohlavek)

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- Question: How to modify the definition of μ-equality, such that we can get the other direction of the HSP theorem?
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Definition

Let $\mathcal{M}$ be a universal algebra of, $\mathcal{L}$ a residuated lattice, $r \in L$, and $\mu : M \longrightarrow L$. The **right** (μ, r) **relation** is the fuzzy relation $\approx^{\mathcal{M}}: M \times M \longrightarrow L$ defined in the following way:

$$(a \approx^{\mathcal{M}} b) =_{def} \begin{cases} \mathbf{1}, & a = b \\ (\mu(a) \vee \mu(b)) \rightarrow r, & a \neq b \end{cases} \tag{3}$$

Example

Let **N** be the set of positive integers, and $\mathcal{M} = \langle N, +, \cdot \rangle$, $r = 1/3$, $\mathcal{L}$ standard lattice on $[0, 1]$ (Lukasiewicz or product structure - Gödel is trivial in this case!),

$$\mu(n) =_{def} \min\left(\frac{1}{3} + \frac{1}{n}, 1\right). \tag{4}$$

If $\approx : N \times N \longrightarrow L$ is the right (μ, r) relation:

$$m \approx n =_{def} \begin{cases} 1, & m = n \\ (\mu(m) \vee \mu(n)) \rightarrow \frac{1}{3} & m \neq n. \end{cases} \tag{5}$$

Then:

- $\mu(n) \in (r, 1]$, for all $m \in N$,
- $\mu(m + n) \leqslant \mu(m)$ and $\mu(m \cdot n) \leqslant \mu(m)$

Right similarity with Lukasiewicz structure

a	b	$\mu(a)$	$\mu(b)$	$a \approx b$
60000	90000	0,33335	0,33334	0,99997
60000	$b > 60000$	0,33335	$\mu(b)$	0,99997
500	60000	0,33533	0,33335	0,998
500	$b > 500$	0,33533	$\mu(b)$	0,998
5	30000	0,53333	0,33337	0,8
5	$b > 5$	0,53333	$\mu(b)$	0,8
3	5	0,66667	0,53333	0,66667
3	$b > 3$	0,66667	$\mu(b)$	0,66667
2	$b > 2$	0,83333	$\mu(b)$	0,5
1	$b > 1$	1	$\mu(b)$	$\frac{1}{3}$

Right similarity with the product structure

a	b	$\mu(a)$	$\mu(b)$	$a \approx b$
60000	90000	0,33335	0,33334	0,99995
60000	$b > 60000$	0,33335	$\mu(b)$	0,99995
500	60000	0,33533	0,33335	0,99404
500	$b > 500$	0,33533	$\mu(b)$	0,99404
5	30000	0,53333	0,33337	0,625
5	$b > 5$	0,53333	$\mu(b)$	0,5
3	5	0,66667	0,53333	0,5
3	$b > 3$	0,66667	$\mu(b)$	0,4
2	$b > 2$	0,83333	$\mu(b)$	0,5
1	$b > 1$	1	$\mu(b)$	$\frac{1}{3}$

Theorem

Let $\mathcal{M}$ be a universal algebra of type Ω, $\mathcal{L}$ a residuated lattice, $r \in L$, $\mu : M \longrightarrow L$, and $\approx^{\mathcal{M}}: M \times M \longrightarrow L$ the right (μ, r) relation. Suppose that $r < \mu(m)$ for all $m \in M$ and for all $f \in \Omega_n$, $n \geqslant 1$, it holds

$$\mu(f^{\mathcal{M}}(m_1, \dots, m_n)) \leqslant \bigwedge_{i=1}^{n} \mu(m_i), \text{ for all } m_1, \dots, m_n \in M, \tag{6}$$

Then $\langle M, \Omega, \approx^{\mathcal{M}} \rangle$ is an $\mathcal{L}-$algebra with fuzzy equality.

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Then $\langle M, \Omega, \approx^{\mathcal{M}} \rangle$ is an $\mathcal{L}-$algebra with fuzzy equality.

RESULTS:

- right (μ, r) identity,
- right (μ, r) equational class of algebras (r is fixed, μ not)
- Birkhoff-like theorems! (HSP stability and equational classes)

Left (μ, s) relation

Definition

Let $\mathcal{M}$ be a universal algebra of, $\mathcal{L}$ a residuated lattice, $s \in L$, and $\mu : M \longrightarrow L$. The **left** (μ, s) **relation** is the fuzzy relation $\approx^{\mathcal{M}}: M \times M \longrightarrow L$ defined in the following way:

$$(a \approx^{\mathcal{M}} b) =_{def} \begin{cases} \mathbf{1}, & a = b \\ (s \rightarrow (\mu(a) \vee \mu(b))), & a \neq b \end{cases} \qquad (7)$$

Example: financial mathematics (money, debt, credits, creditworthiness, solvency,...)

Theorem

Let $\mathcal{M}$ be a universal algebra of type Ω, $\mathcal{L}$ a residuated lattice, $s \in L$, $\mu : M \longrightarrow L$, and $\approx^{\mathcal{M}}: M \times M \longrightarrow L$ the left (μ, s) relation. Suppose that $\mu(m) < s$ for all $m \in M$ and for all $f \in \Omega_n$, $n \geqslant 1$, it holds

$$\bigvee_{i=1}^{n} \mu(m_i) \leqslant \mu(f^{\mathcal{M}}(m_1, \ldots, m_n) \text{ for all } m_1, \ldots, m_n \in M,$$

Then $\langle M, \Omega, \approx^{\mathcal{M}} \rangle$ is an $\mathcal{L}-$algebra with fuzzy equality.

Theorem

Let $\mathcal{M}$ be a universal algebra of type Ω, $\mathcal{L}$ a residuated lattice, $s \in L$, $\mu : M \longrightarrow L$, and $\approx^{\mathcal{M}}: M \times M \longrightarrow L$ the left (μ, s) relation. Suppose that $\mu(m) < s$ for all $m \in M$ and for all $f \in \Omega_n$, $n \geqslant 1$, it holds

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Then $\langle M, \Omega, \approx^{\mathcal{M}} \rangle$ is an $\mathcal{L}-$algebra with fuzzy equality.

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Further directions

- properties of special equational classes
- back to groupoids!
- structure of truth values as parameter! (how this impact the results)
- applications?

Thank you for your attention!