Algebras of Fuzzy Sets

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Join work with I. Bošnjak, G. Vojvodić, M. Bradić

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FSTA 2012, Liptovský Ján
Subject of this talk

- universal-algebraic investigations of fuzzy structures
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- lattice of idempotent fuzzy subsets of a groupoid
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- generalization: algebras of fuzzy sets (fuzzy power algebras)-results about homomorphisms, subalgebras, direct products
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- universal-algebraic investigations of fuzzy structures
- lattice of idempotent fuzzy subsets of a groupoid
- generalization: algebras of fuzzy sets (fuzzy power algebras)- results about homomorphisms, subalgebras, direct products
- very new results: special kinds of fuzzy equalities, identities, equational classes, Birkhoff-style theorems
The beginnings

- coming from universal algebra, power structures...
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- $\mathcal{F}_2(D)$ of all idempotent fuzzy subsets of a cancellative semigroup $D$ forms a complete lattice
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- $\mathcal{F}_2(D)$ of all idempotent fuzzy subsets of a cancellative semigroup $D$ forms a complete lattice
- results concerning the set $\mathcal{F}(D)$ of all fuzzy subsets of $D$ and its substructures $\mathcal{F}_{-1}(D)$ and $\mathcal{F}_2(D)$. 
The beginnings

- coming from universal algebra, power structures...
- $\mathcal{F}_2(D)$ of all idempotent fuzzy subsets of a cancellative semigroup $D$ forms a complete lattice
- results concerning the set $\mathcal{F}(D)$ of all fuzzy subsets of $D$ and its substructures $\mathcal{F}_{-1}(D)$ and $\mathcal{F}_2(D)$.
- If $D$ is a cancellative groupoid, then the sup-min product is distributive over an arbitrary intersection of fuzzy sets in $\mathcal{F}(D)$!
Lattice of fuzzy sets of a groupoid

- $\mathcal{D} = \langle D, \cdot \rangle$ a groupoid, $\mathcal{F}(D)$ all fuzzy subsets of $D$, $\mu : D \to [0, 1]$
Lattice of fuzzy sets of a groupoid

- \( \mathcal{D} = \langle D, \cdot \rangle \) a groupoid, \( \mathcal{F}(D) \) all fuzzy subsets of \( D \),
  \( \mu : D \to [0, 1] \)
- sup-min *product* of two fuzzy subsets \( \lambda \) and \( \mu \):
  \[
  (\lambda \cdot \mu)(x) = \sup_{x=ab} \min(\lambda(a), \mu(b)).
  \]
Lattice of fuzzy sets of a groupoid

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- sup-min *product* of two fuzzy subsets $\lambda$ and $\mu$:

\[(\lambda \cdot \mu)(x) = \sup_{x=ab} \min(\lambda(a), \mu(b)).\]

- $\mathcal{F}_{-1}(D) = \{ \lambda \in \mathcal{F}(D) \mid \lambda \subseteq \lambda \cdot \lambda \}$

- $\mathcal{F}_2(D) = \{ \lambda \in \mathcal{F}(D) \mid \lambda = \lambda \cdot \lambda \}$
RID and LID groupoids

Definition

For a groupoid $G$ we will say that it is right intersection-distributive (RID) if for any family $\{\lambda_i \mid i \in I\} \subseteq \mathcal{F}(D)$ and any $\mu \in \mathcal{F}(D)$ it holds

$$\left( \bigcap_{i \in I} \lambda_i \right) \cdot \mu = \bigcap_{i \in I} (\lambda_i \cdot \mu).$$

Similarly, $D$ is left intersection-distributive (LID).
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- The only RID semigroups are left-zero semigroups. The only LID semigroups are...
- A finite groupoid can not be both RID and LID.
- An infinite groupoid $D$ is both RID and LID iff in its Cayley table every element from $D$ appears at most once.
- There are infinite groupoids which are at the same time RID and LID.
Theorem

(1) A groupoid $D$ is RID iff $D$ satisfies the quasiidentity

$$xy = zt \Rightarrow x = z.$$ 

(2) A groupoid $D$ is LID iff $D$ satisfies the quasiidentity

$$yx = tz \Rightarrow x = z.$$
Results

Theorem

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(2) A groupoid $D$ is LID iff $D$ satisfies the quasiidentity

$$yx = tz \Rightarrow x = z.$$ 

Theorem

Let $D$ be a groupoid. Then $F_2(D)$ is a complete lattice which is a complete join-sublattice of $F_1(D)$, and a complete meet-sublattice of $F^{-1}(D)$. 

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Algebras of Fuzzy Sets
General problem: fuzzy power algebras

Universal algebras in fuzzy world

- Rosenfeld (1971) - fuzzy groups
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Universal algebras in fuzzy world

- Rosenfeld (1971) - fuzzy groups
- Vojvodic, Šešelja (1983), Tepavčević
Different approaches

- *different degree of generality*: first concrete algebras (fuzzy groups, lattices, vector spaces,...) then unifying results (homomorphisms, congruences, factor algebras,...)
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• *different fuzzyfication*: what is fuzzyfied - the universe, operations, equality relation,...

• We choose: universal algebras, complete residuated lattices (sometimes, complete lattices), and trying to connect different fuzzyfifications...
Residuated lattices

A *residuated lattice* is an algebra \( L = \langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow \rangle \) where

1. \( \langle L, \wedge, \vee, 0, 1, \rangle \) is a lattice with the least element 0 and the greatest element 1.
A residuated lattice is an algebra $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, \otimes, \to \rangle$ where

(i) $\langle L, \wedge, \vee, 0, 1, \rangle$ is a lattice with the least element 0 and the greatest element 1.

(ii) $\langle L, \otimes, 1 \rangle$ is a commutative monoid.
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Residuated lattices

A *residuated lattice* is an algebra $\mathcal{L} = \langle L, \land, \lor, 0, 1, \otimes, \to \rangle$ where

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If the lattice $\langle L, \land, \lor, 0, 1, \rangle$ is complete, then $\mathcal{L}$ is a complete residuated lattice.
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- examples: on $[0, 1]$, Lukasiewicz, Gödel and product structures
Basic notions and notation

Let $\mathcal{L} = \langle L, \land, \lor, 0, 1, \otimes, \rightarrow \rangle$ be a complete residuated lattice, $X$ a nonempty set.

- set of all $L$-fuzzy sets on $X$: $\mathcal{F}_{\mathcal{L}}(X)$, or $\mathcal{F}(X)$ or $L^X$
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- **fuzzy relation on $A$:** $\eta : A^n \to L$.
- **fuzzy equivalence:** a binary fuzzy relation on $A$ which is
  - reflexive: $\eta(x, x) = 1$, for all $x \in A$,
  - symmetric: $\eta(x, y) = \eta(y, x)$, for $x, y \in A$,
  - transitive: $\eta(x, y) \otimes \eta(y, z) \leq \eta(x, z)$, for all $x, y, z, \in A$. 

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  - transitive: $\eta(x, y) \otimes \eta(y, z) \leq \eta(x, z)$, for all $x, y, z, \in A$.
- **fuzzy equality:** fuzzy equivalence relation $\eta$ if from $\eta(x, y) = 1$ it follows $x = y$. 

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Algebras of Fuzzy Sets
Extension Principle

- extension principle: how to extend a function
  \( f : X_1 \times \cdots \times X_n \rightarrow Y \) into \( \tilde{f} : L^{X_1} \times \cdots \times L^{X_n} \rightarrow L^Y \) (L. Zadeh)
- applications: fuzzy arithmetic, engineering problems, analysis of
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- applications: fuzzy arithmetic, engineering problems, analysis of discrete dynamical systems, fuzzy fractals, fuzzy transportation problems...
- Let \( \mathcal{L} \) be a complete lattice or a complete residuated lattice,
  \( f : X_1 \times \cdots \times X_n \rightarrow Y \). Define \( \bar{f} : L^{X_1} \times \cdots \times L^{X_n} \rightarrow L^Y \)
  
  \[
  \bar{f}(\mu_1, \ldots, \mu_n)(y) = \bigvee_{x_i \in X_i, f(x_1, \ldots, x_n) = y} \mu_1(x_1) \land \cdots \land \mu_n(x_n). \quad (1)
  \]
If we have a residuated lattice, $f : X_1 \times \cdots \times X_n \rightarrow Y$, we can extend it to a function $\bar{f} : L^{X_1} \times \cdots \times L^{X_n} \rightarrow L^Y$ in an alternative way:

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\bar{f}^{\otimes}(\mu_1, \ldots, \mu_n)(y) = \bigvee_{x_i \in X_i} \mu_1(x_1) \otimes \cdots \otimes \mu_n(x_n).
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• extension principle in the context of algebras:
  • Murali (1991); set of truth values: real unit interval (min/max)
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Algebra of fuzzy sets 1

Definition

Let \( \mathcal{L} \) be a complete lattice or a residuated lattice and \( \mathcal{A} = \left\langle A, \{f \mid f \in \Omega\} \right\rangle \) be a universal algebra. If \( f \in \Omega \) is an \( n \)-ary fundamental operation of \( \mathcal{A} \), define \( \bar{f}^\land : \mathcal{F}(A)^n \to \mathcal{F}(A) \) in the following way:

\[
\bar{f}^\land(\mu_1, \ldots, \mu_n)(y) = \bigvee_{\substack{x_i \in X_i \\text{ s.t. } f(x_1, \ldots, x_n) = y}} \mu_1(x_1) \land \cdots \land \mu_n(x_n).
\]

The algebra \( \mathcal{F}^\land(\mathcal{A}) = \left\langle \mathcal{F}(A), \{\bar{f}^\land \mid f \in \Omega\} \right\rangle \) will be called the \( \land \)-algebra of fuzzy sets induced by \( \mathcal{A} \).
Definition

Let \( \mathcal{L} \) be a residuated lattice and \( \mathcal{A} = \langle A, \{f \mid f \in \Omega \} \rangle \) be a universal algebra. If \( f \in \Omega \) is an \( n \)-ary fundamental operation of \( \mathcal{A} \), define \( \bar{f}^\otimes : \mathcal{F}(A)^n \to \mathcal{F}(A) \) in the following way:

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The algebra \( \mathcal{F}^\otimes(\mathcal{A}) = \langle \mathcal{F}(A), \{\bar{f}^\otimes \mid f \in \Omega \} \rangle \) will be called the \( \otimes \)-algebra of fuzzy sets induced by \( \mathcal{A} \).
Definition

Let $\mathcal{L}$ be a residuated lattice and $\mathcal{A} = \langle A, \{f \mid f \in \Omega\} \rangle$ be a universal algebra. If $f \in \Omega$ is an $n$-ary fundamental operation of $\mathcal{A}$, define $\tilde{f}^{\otimes} : \mathcal{F}(A)^n \to \mathcal{F}(A)$ in the following way:

$$\tilde{f}^{\otimes}(\mu_1, \ldots, \mu_n)(y) = \bigvee_{\substack{x_i \in x_i \\ f(x_1, \ldots, x_n) = y}} \mu_1(x_1) \otimes \cdots \otimes \mu_n(x_n).$$

The algebra $\mathcal{F}^{\otimes}(\mathcal{A}) = \langle \mathcal{F}(A), \{\tilde{f}^{\otimes} \mid f \in \Omega\} \rangle$ will be called the $\otimes$-algebra of fuzzy sets induced by $\mathcal{A}$.

Of course, if $\mathcal{L}$ is a complete Heyting algebra, or more specially, if $L = \{0, 1\}$, the two kinds of induced algebras of fuzzy sets coincide.
Power algebras 1

- crisp case: both kinds of induced algebras of fuzzy sets become the ordinary *power algebra* of $A$ ( *algebra of complexes* or *global* of $A$ ).
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- crisp case: both kinds of induced algebras of fuzzy sets become the ordinary *power algebra* of $A$ (*algebra of complexes* or *global* of $A$).

**Definition**

Let $A$ be a non-empty set, $\mathcal{P}(A)$ the set of all subsets of $A$, and $f : A^n \to A$. We define $f^+ : \mathcal{P}(A)^n \to \mathcal{P}(A)$ in the following way:

$$f^+ (X_1, \ldots, X_n) = \{ f(x_1, \ldots, x_n) \mid x_1 \in X_1, \ldots, x_n \in X_n \}.$$  

If $\mathcal{A} = \langle A, \{f \mid f \in \Omega\} \rangle$ is an algebra, the **power algebra** (or **complex algebra**, or **global**) $\mathcal{P}(\mathcal{A})$ is defined as:

$$\mathcal{P}(\mathcal{A}) = \langle \mathcal{P}(A), \{f^+ \mid f \in \Omega\} \rangle.$$
Power algebras 2

- crisp power algebras are used:
  - group theory, semigroup theory
  - lattices (the set of ideals of a distributive lattice $L$ again forms a lattice, and meets and joins in the new lattice are precisely the power operations of meets and joins in $L$)
  - formal language theory (the product of two languages is simply the power operation of concatenation of words)
  - non-classical logics (Jonsson, Tarski, Boolean algebras with operators...)

Proposition

Let $A$ be any universal algebra, and $L$ the usual two element Boolean algebra. Then both of $\wedge$-algebra and the $\otimes$-algebra of $L$-fuzzy sets induced by $A$ coincide with the power algebra of $A$, i.e.

$$F^{\wedge}(A) = F^{\otimes}(A) = P(A).$$
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  - formal language theory (the product of two languages is simply the power operation of concatenation of words)
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**Proposition**

Let $\mathcal{A}$ be any universal algebra, and $\mathcal{L}$ the usual two element Boolean algebra. Then both of $\land$-algebra and the $\otimes$-algebra of $L$-fuzzy sets induced by $\mathcal{A}$ coincide with the power algebra of $\mathcal{A}$, i.e.

$$\mathcal{F}^\land(\mathcal{A}) = \mathcal{F}^\otimes(\mathcal{A}) = \mathcal{P}(\mathcal{A}).$$
Homomorphisms 1

**Definition**

Let $A$ and $B$ be algebras of the same type $\Omega$. A mapping $\alpha : A \to B$ is a **homomorphism from** $A$ to $B$ if for all $n \geq n$, all $f \in \Omega_n$, all $a_1, a_2, \ldots, a_n \in A$,

$$\alpha(f^A(a_1, a_2, \ldots, a_n)) = f^B(\alpha(a_1), \alpha(a_2), \ldots, \alpha(a_n)).$$

**Proposition**

Let $\mathcal{L}$ be a lattice or a residuated lattice. Then:

(a) If $\alpha : A \to B$, $\beta : B \to C$, then $(\beta \circ \alpha) = \overline{\beta} \circ \overline{\alpha}$.

(b) If $\alpha : A \to B$ is a bijection, then $\overline{\alpha} : \mathcal{F}(A) \to \mathcal{F}(B)$ is also a bijection.
Let $\alpha : A \to B$ be a homomorphism. Will the induced mapping $\overline{\alpha} : \mathcal{F}(A) \to \mathcal{F}(B)$ be a homomorphism from $\mathcal{F}^\otimes(A)$ to $\mathcal{F}^\otimes(B)$, and from $\mathcal{F}^\wedge(A)$ to $\mathcal{F}^\wedge(B)$? The two kinds of induced algebras of fuzzy sets do not behave in the same way!
Homomorphisms 2

Let $\alpha : A \rightarrow B$ be a homomorphism. Will the induced mapping $\overline{\alpha} : F(A) \rightarrow F(B)$ be a homomorphism from $F \otimes (A)$ to $F \otimes (B)$, and from $F^{\wedge} (A)$ to $F^{\wedge} (B)$? The two kinds of induced algebras of fuzzy sets do not behave in the same way!

Theorem

Let $L$ be a complete residuated lattice, $A$ and $B$ two algebras of type $\Omega$. If $\alpha : A \rightarrow B$ is a homomorphism, then $\overline{\alpha} : F \otimes (A) \rightarrow F \otimes (B)$ is also a homomorphism.

BUT: there are algebras $A$, $B$, a complete lattice $L$, such that $\alpha : A \rightarrow B$ is a homomorphism, but $\overline{\alpha} : F^{\wedge} (A) \rightarrow F^{\wedge} (B)$ is not a homomorphism!
**Example**

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Let \(\mathcal{L}\) be the pentagon (with \(0 < r < q < 1, 0 < p < 1, p\) not being comparable to \(r\) or \(q\)). There is a homomorphism from \(\mathcal{A}\) to \(\mathcal{B}\) such that \(\bar{\alpha} : \mathcal{F}^\wedge(\mathcal{A}) \to \mathcal{F}^\wedge(\mathcal{B})\) is not a homomorphism.
Theorem

Let $\mathcal{L}$ be a completely distributive lattice, $\mathcal{A}$ and $\mathcal{B}$ two algebras of type $\Omega$. If $\alpha : \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism, then $\overline{\alpha} : \mathcal{F}^\wedge(\mathcal{A}) \rightarrow \mathcal{F}^\wedge(\mathcal{B})$ is also a homomorphism.
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Theorem

Let $\mathcal{L}$ be a complete lattice or a complete residuated lattice, and $\mathcal{A}$ a subalgebra of algebra $\mathcal{B}$. Then $\mathcal{F}^\wedge(\mathcal{A})$ can be embedded into the algebra $\mathcal{F}^\wedge(\mathcal{B})$. 
Theorem

Let $\mathcal{L}$ be a completely distributive lattice, $\mathcal{A}$ and $\mathcal{B}$ two algebras of type $\Omega$. If $\alpha : \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism, then $\overline{\alpha} : F^\wedge(\mathcal{A}) \rightarrow F^\wedge(\mathcal{B})$ is also a homomorphism.

Theorem

Let $\mathcal{L}$ be a complete lattice or a complete residuated lattice, and $\mathcal{A}$ a subalgebra of algebra $\mathcal{B}$. Then $F^\wedge(\mathcal{A})$ can be embedded into the algebra $F^\wedge(\mathcal{B})$.

Theorem

Let $\mathcal{L}$ be a complete residuated lattice, and $\mathcal{A}$ a subalgebra of algebra $\mathcal{B}$. Then $F\otimes(\mathcal{A})$ can be embedded into the algebra $F\otimes(\mathcal{B})$. 

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Algebras of Fuzzy Sets
**Fuzzy products 1**

**Definition**

The **fuzzy product** of the family of fuzzy subsets \( \langle \eta_i \in \mathcal{F}(A_i) \mid i \in I \rangle \) is the fuzzy subset \( \prod^\wedge \eta_i : \prod A_i \to L \) defined in the following way: if \( x \in \prod A_i \), where \( x_i = x(i) \) for \( i \in I \), then

\[
(\prod^\wedge \langle \eta_i \mid i \in I \rangle)(x) = \bigwedge \{ \eta_i(x_i) \mid i \in iI \}.
\]
Fuzzy products 1

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\[
(\prod^\wedge \langle \eta_i \mid i \in I \rangle)(x) = \bigwedge \{ \eta_i(x_i) \mid i \in iI \}.
\]

Theorem

Let \( \mathcal{L} \) be a completely distributive lattice and \( \langle A_i \mid i \in I \rangle \) be a family of algebras of the type \( \Omega \). The mapping

\[
\varphi : \prod_{i \in I} \mathcal{F}(A_i) \to \mathcal{F}(\prod_{i \in I} A_i)
\]

defined by \( \varphi(\eta) = \prod^\wedge \eta(i) \) is a homomorphism from \( \prod_{i \in I} \mathcal{F}(^\wedge A_i) \) to \( \mathcal{F}(\prod_{i \in I} A_i) \).
Fuzzy products 2

It is not hard to see that the above defined mapping
\[ \varphi : \prod_{i \in I} \mathcal{F}(A_i) \rightarrow \mathcal{F}(\prod_{i \in I} A_i) \]
is not necessarily injective.

Example

Take \( L = [0, 1] \), \( I = \{1, 2, 3\} \) and \( \eta_1, \eta_2, \eta_3, \eta_4 : A \rightarrow L \) such that
\[
\eta_1(x) = 0.1, \quad \text{for all } x \in A \\
\eta_2(x) = 0.1, \quad \text{for all } x \in A \\
\eta_3(x) = 0.9, \quad \text{for all } x \in A \\
\eta_4(x) = 0.8, \quad \text{for all } x \in A
\]
Then \( \varphi((\eta_1, \eta_2, \eta_3)) = \varphi((\eta_1, \eta_2, \eta_4)) = \mu \), where \( \mu(x) = 0.1 \) for all \( x \in A \).
Fuzzy products 3

**Definition**

\[ \mathcal{F}_+(A) \subseteq \mathcal{F}(A) \] is defined by

\[ \mathcal{F}_+(A) = \{ \eta : A \rightarrow L \mid (\exists x \in A) \, \eta(x) = 1 \}. \]
Definition

$\mathcal{F}_+(A) \subseteq \mathcal{F}(A)$ is defined by
$\mathcal{F}_+(A) = \{ \eta : A \to L \mid (\exists x \in A) \; \eta(x) = 1 \}$.

Theorem

Let $\langle A_i \mid i \in I \rangle$ be a family of sets. Then the mapping
$\varphi_+ : \prod_{i \in I} \mathcal{F}_+(A_i) \to \mathcal{F}_+(\prod_{i \in I} A_i)$ defined by $\varphi_+(\eta) = \prod_{i \in I}^\wedge \eta(i)$ is injective.
Definition

\( \mathcal{F}_+(A) \subseteq \mathcal{F}(A) \) is defined by
\[
\mathcal{F}_+(A) = \{ \eta : A \to L \mid (\exists x \in A) \, \eta(x) = 1 \}.
\]

Theorem

Let \( \langle A_i \mid i \in I \rangle \) be a family of sets. Then the mapping
\[
\varphi_+ : \prod_{i \in I} \mathcal{F}_+(A_i) \to \mathcal{F}_+(\prod_{i \in I} A_i)
\]
defined by \( \varphi_+(\eta) = \prod_{i \in I} \eta(i) \) is injective.

Theorem

Let \( \mathcal{A} \) be an algebra of type \( \Omega \). Then \( \mathcal{F}_+(A) \) is a subuniverse of the algebras \( \mathcal{F}^\wedge(A) \) and \( \mathcal{F}^\otimes(A) \).
It can be proved that the two kind of algebras of fuzzy sets behave in different way in respect to direct products.

**Theorem**

Let $\mathcal{L}$ be a completely distributive lattice and $\langle A_i \mid i \in I \rangle$ be a family of algebras of the type $\Omega$. The mapping

$$\varphi : \prod_{i \in I} \mathcal{F}(A_i) \to \mathcal{F}(\prod_{i \in I} A_i)$$

defined by $\varphi(\eta) = \prod^\wedge \eta(i)$ is a homomorphism from $\prod_{i \in I} \mathcal{F}^\wedge(A_i)$ to $\mathcal{F}^\wedge(\prod_{i \in I} A_i)$.
Fuzzy products 4

It can be proved that the two kind of algebras of fuzzy sets behave in different way in respect to direct products.

**Theorem**

Let \( \mathcal{L} \) be a completely distributive lattice and \( \langle \mathcal{A}_i \mid i \in I \rangle \) be a family of algebras of the type \( \Omega \). The mapping
\[
\varphi : \prod_{i \in I} \mathcal{F}(A_i) \to \mathcal{F}(\prod_{i \in I} A_i)
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defined by \( \varphi(\eta) = \prod^\wedge \eta(i) \) is a homomorphism from \( \prod_{i \in I} \mathcal{F}^\wedge(A_i) \) to \( \mathcal{F}^\wedge(\prod_{i \in I} A_i) \).

**BUT:** We can construct two algebras \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) such that the mapping \( \varphi : \mathcal{F}(A_1) \times \mathcal{F}(A_2) \to \mathcal{F}(A_1 \times A_2) \) defined by
\[
\varphi(\langle \eta, \mu \rangle) = \prod_{i \in \{1,2\}}^\wedge \eta(i)
\]
is not a homomorphism from \( \mathcal{F}^\otimes(\mathcal{A}_1) \times \mathcal{F}^\otimes(\mathcal{A}_2) \) to \( \mathcal{F}^\otimes(\mathcal{A}_1 \times \mathcal{A}_2) \).
Example

The algebras $A_1$ and $A_2$ will be groupoids given respectively by their Cayley’s tables:

\[
\begin{array}{c|cc}
  \cdot & a & b \\
\hline
  a & b & a \\
  b & b & b \\
\end{array}
\quad
\begin{array}{c|ccc}
  \cdot & c & d \\
\hline
  c & c & d \\
  d & c & c \\
\end{array}
\]

Let $\mathcal{L}$ be standard residual lattice on the real unit interval $[0,1]$ with the product structure, i.e. $\otimes$ is the usual product of real numbers. Let us take $\eta = \langle \eta_1, \eta_2 \rangle$, $\mu = \langle \mu_1, \mu_2 \rangle$, where $\eta_1(a) = 0.6$, $\eta_2(c) = 0.5$, $\mu_1(b) = 0.7$, $\mu_2(d) = 0.8$. It can be proved that

\[
(\varphi(\eta \cdot \mu))(\langle a, d \rangle) \neq (\varphi(\eta) \cdot \varphi(\mu))(\langle a, d \rangle).
\]
Algebras with fuzzy equalities

Proposition

Let $\mathcal{L}$ be a complete residuated lattice, $\mathcal{A}$ a universal algebra, and $\approx$ the similarity relation defined on $\mathcal{F}(\mathcal{A}) = \mathcal{L}^\mathcal{A}$ by

$$(\eta \approx \mu) = \wedge \{ \eta(x) \leftrightarrow \mu(x) \mid x \in A \}.$$ 

Then the structures $\langle \mathcal{F}(\mathcal{A}), \approx \rangle$ and $\langle \mathcal{F}^\otimes(\mathcal{A}), \approx \rangle$ are algebras with fuzzy equality.
Algebras with fuzzy equalities

Proposition

Let $\mathcal{L}$ be a complete residuated lattice, $\mathcal{A}$ a universal algebra, and $\approx$ the similarity relation defined on $\mathcal{F}(\mathcal{A}) = \mathcal{L}^\mathcal{A}$ by

$$(\eta \approx \mu) = \wedge \{ \eta(x) \leftrightarrow \mu(x) \mid x \in \mathcal{A} \}.$$ 

Then the structures $\langle \mathcal{F}^\wedge(\mathcal{A}), \approx \rangle$ and $\langle \mathcal{F}^\otimes(\mathcal{A}), \approx \rangle$ are algebras with fuzzy equality.

Definition (Belohlavek)

Let $\mathcal{L}$ be a complete residuated lattice, $\langle \mathcal{A}, \Omega \rangle$ be a universal algebra, and $\approx$ an fuzzy equality on $\mathcal{A}$. Then the structure $\mathcal{A} = \langle \mathcal{A}, \Omega, \approx \rangle$ is an $\mathcal{L}$-algebra with fuzzy equality if each operation $f \in \Omega$ is compatible with $\approx$, i.e. for any $n$-ary $f \in \Omega$, for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$ we have

$$(a_1 \approx b_1) \otimes \cdots \otimes (a_n \approx b_n) \leq f(a_1, \ldots, a_n) \approx f(b_1, \ldots, b_n).$$
Fuzzy (sub)algebras and equalities

Definition (B. Šešelja, A. Tepavčević)

Let $\mathcal{A}$ be a universal algebra of type $\Omega$, $L$ a complete lattice, $\mu : \mathcal{A} \to L$ a fuzzy (sub)algebra of $\mathcal{A}$. A $\mu$-fuzzy equality is any binary fuzzy relation $E : \mathcal{A}^2 \to L$ such that:

- $E(x, y) < \mu(x) = E(x, x)$, for all different $x, y \in \mathcal{A}$,
- $E(x, y) = E(y, x)$, for all $x, y \in \mathcal{A}$,
- $E(x, y) \land E(y, z) \leq E(x, z)$, for all $x, y, z \in \mathcal{A}$,
- $E(a_1, b_1) \land E(a_2, b_2) \land \cdots \land E(a_n, b_n) \leq E(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n))$, for all...
Fuzzy (sub)algebras and equalities

**Definition (B. Šešelja, A. Tepavčević)**

Let $A$ be a universal algebra of type $\Omega$, $L$ a complete lattice, $\mu : A \to L$ a fuzzy (sub)algebra of $A$. A $\mu$-**fuzzy equality** is any binary fuzzy relation $E : A^2 \to L$ such that:

- $E(x, y) < \mu(x) = E(x, x)$, for all different $x, y \in A$,
- $E(x, y) = E(y, x)$, for all $x, y \in A$,
- $E(x, y) \land E(y, z) \leq E(x, z)$, for all $x, y, z \in A$,
- $E(a_1, b_1) \land E(a_2, b_2) \land \cdots \land E(a_n, b_n) \leq E(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n))$, for all...

Let $A$ be a universal algebra of type $\Omega$, $L$ a complete lattice. Then $\mu : A \to L$ is a **fuzzy (sub)algebra** of $A$ if for all...

$$\mu(a_1) \land \mu(a_2) \land \cdots \land \mu(a_n) < \mu(f(a_1, a_2, \ldots, a_n)).$$
Right \((\mu, r)\) relation

- algebras with fuzzy equality: variety theorem (Belohlavek)
Right $(\mu, r)$ relation

- algebras with fuzzy equality: variety theorem (Belohlavek)
- fuzzy (sub)algebras with fuzzy equality: one direction of variety theorem

Definition
Let $M$ be a universal algebra of, $L$ a residuated lattice, $r \in L$, and $\mu: M \to L$. The right $(\mu, r)$ relation is the fuzzy relation $\approx: M \times M \to L$ defined in the following way:

$$(a \approx b) = \defrule \begin{cases} 1, & a = b \\mu(a) \lor \mu(b) \rightarrow r, & a \neq b \end{cases}$$
Right \((\mu, r)\) relation

- algebras with fuzzy equality: variety theorem (Belohlavek)
- fuzzy (sub)algebras with fuzzy equality: one direction of variety theorem
- Question: How to modify the definition of \(\mu\)-equality, such that we can get the other direction of the HSP theorem?
Right \((\mu, r)\) relation

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- Milanka Bradić (PhD student): a new definition of \(\mu\)-equality...
Right \((\mu, r)\) relation

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- Milanka Bradić (PhD student): a new definition of \(\mu\)-equality...

**Definition**

Let \(\mathcal{M}\) be a universal algebra of, \(\mathcal{L}\) a residuated lattice, \(r \in L\), and \(\mu : M \rightarrow L\). The **right (\(\mu, r\)) relation** is the fuzzy relation \(\approx^{\mathcal{M}} : M \times M \rightarrow L\) defined in the following way:

\[
(a \approx^{\mathcal{M}} b) \overset{\text{def}}{=} \begin{cases} 
1, & a = b \\
(\mu(a) \lor \mu(b)) \rightarrow r, & a \neq b
\end{cases}
\]
Example

Let \( \mathbb{N} \) be the set of positive integers, and \( \mathcal{M} = \langle \mathbb{N}, +, \cdot \rangle, \ r = 1/3, \mathcal{L} \) standard lattice on \([0, 1]\) (Lukasiewicz or product structure - Gödel is trivial in this case!),

\[
\mu(n) = \text{def} \ \min \left( \frac{1}{3} + \frac{1}{n}, 1 \right).
\]  

(4)

If \( \approx : \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{L} \) is the right \((\mu, r)\) relation:

\[
m \approx n = \text{def} \begin{cases} 1, & m = n \\ \mu(m) \lor \mu(n) \rightarrow \frac{1}{3} & m \neq n. \end{cases}
\]  

(5)

Then:

- \( \mu(n) \in (r, 1], \) for all \( m \in \mathbb{N}, \)
- \( \mu(m + n) \leq \mu(m) \) and \( \mu(m \cdot n) \leq \mu(m) \)
Right similarity with Lukasiewicz structure

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<tr>
<th>$a$</th>
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<th>$\mu(a)$</th>
<th>$\mu(b)$</th>
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Right similarity with the product structure

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Theorem

Let $\mathcal{M}$ be a universal algebra of type $\Omega$, $\mathcal{L}$ a residuated lattice, $r \in L$, $\mu : M \rightarrow L$, and $\approx^\mathcal{M} : M \times M \rightarrow L$ the right $(\mu, r)$ relation. Suppose that $r < \mu(m)$ for all $m \in M$ and for all $f \in \Omega_n$, $n \geq 1$, it holds

$$\mu(f^\mathcal{M}(m_1, \ldots, m_n)) \leq \bigwedge_{i=1}^{n} \mu(m_i), \text{ for all } m_1, \ldots, m_n \in M,$$

(6)

Then $\langle M, \Omega, \approx^\mathcal{M} \rangle$ is an $\mathcal{L}$–algebra with fuzzy equality.
Theorem

Let $\mathcal{M}$ be a universal algebra of type $\Omega$, $\mathcal{L}$ a residuated lattice, $r \in L$, $\mu : \mathcal{M} \to L$, and $\approx^\mathcal{M} : \mathcal{M} \times \mathcal{M} \to L$ the right $(\mu, r)$ relation. Suppose that $r < \mu(m)$ for all $m \in \mathcal{M}$ and for all $f \in \Omega_n$, $n \geq 1$, it holds

$$\mu(f^\mathcal{M}(m_1, \ldots, m_n)) \leq \bigwedge_{i=1}^n \mu(m_i), \text{ for all } m_1, \ldots, m_n \in \mathcal{M}, \quad (6)$$

Then $\langle \mathcal{M}, \Omega, \approx^\mathcal{M} \rangle$ is an $\mathcal{L}$—algebra with fuzzy equality.

RESULTS:

- right $(\mu, r)$ identity,
- right $(\mu, r)$ equational class of algebras ($r$ is fixed, $\mu$ not)
- Birkhoff-like theorems! (HSP stability and equational classes)
Left \((\mu, s)\) relation

**Definition**

Let \(M\) be a universal algebra of, \(L\) a residuated lattice, \(s \in L\), and \(\mu : M \rightarrow L\). The **left \((\mu, s)\) relation** is the fuzzy relation \(\approx^M : M \times M \rightarrow L\) defined in the following way:

\[
(a \approx^M b) = \begin{cases} 
1, & a = b \\
(s \rightarrow (\mu(a) \lor \mu(b))), & a \neq b 
\end{cases}
\]  

(7)

**Example:** financial mathematics (money, debt, credits, creditworthiness, solvency,...)
Theorem

Let $\mathcal{M}$ be a universal algebra of type $\Omega$, $\mathcal{L}$ a residuated lattice, $s \in L$, $\mu : M \rightarrow L$, and $\approx^\mathcal{M} : M \times M \rightarrow L$ the left $(\mu, s)$ relation. Suppose that $\mu(m) < s$ for all $m \in M$ and for all $f \in \Omega_n$, $n \geq 1$, it holds

$$\bigvee_{i=1}^n \mu(m_i) \leq \mu(f^\mathcal{M}(m_1, \ldots, m_n)) \text{ for all } m_1, \ldots, m_n \in M,$$

Then $\langle \mathcal{M}, \Omega, \approx^\mathcal{M} \rangle$ is an $\mathcal{L}$—algebra with fuzzy equality.
Let $\mathcal{M}$ be a universal algebra of type $\Omega$, $\mathcal{L}$ a residuated lattice, $s \in L$, $\mu : M \rightarrow L$, and $\approx^\mathcal{M} : M \times M \rightarrow L$ the left $(\mu, s)$ relation. Suppose that $\mu(m) < s$ for all $m \in M$ and for all $f \in \Omega_n$, $n \geq 1$, it holds

$$\bigvee_{i=1}^n \mu(m_i) \leq \mu(f^\mathcal{M}(m_1, \ldots, m_n))$$ for all $m_1, \ldots, m_n \in M$.

Then $\langle M, \Omega, \approx^\mathcal{M} \rangle$ is an $\mathcal{L} -$ algebra with fuzzy equality.

RESULTS:

- left $(\mu, s)$ identity,
- left $(\mu, s)$ equational class of algebras ($r$ is fixed, $\mu$ not)
- Birkhoff-like theorems! (HSP stability and equational classes)
Further directions

- properties of special equational classes
- back to groupoids!
- structure of truth values as parameter! (how this impact the results)
- applications?
Thank you for your attention!