IT4Innovations
Centre of Excellence

Similarities and differences of induced (set-valued and fuzzy) discrete dynamical systems

Jiří Kupka

Centre of Excellence IT4Innovations - Division University of Ostrava Institute for Research and Applications of Fuzzy Modeling
University of Ostrava
Czech Republic
Jiri.Kupka@osu.cz

December 1, 2011

EVROPSKÁ UNIE
EVROPSKÝ FOND PRO REGIONÁLNÍ ROZVOJ
INVESTICE DO VAŠÍ BUDOUCNOSTI

OP Výzkum a vývoj pro inovace

Definition

X ... be a compact metric space

$\varphi : X \to X$ ... a continuous map ($\varphi \in C(X)$)

Then a pair (X, φ) form a **discrete dynamical system.**

Basic notions:

- n-th **iteration** of $x \in X$ is defined inductively by

$$\varphi^0(x) = x,\ \varphi^{n+1}(x) = \varphi(\varphi^n(x))$$

- $\{\varphi^n(x)\}_{n\in\mathbb{N}}$ ... a **trajectory**
- $\varphi(x) = x$ ... x is a **fixed** point
- $\varphi^p(x) = x$ ... x is a **periodic** point
- $y \in \omega_\varphi(x)$ ... y is an ω-**limit** point

Example.

$$X := I = [0,1],\ \varphi(x) = x^2.$$

Then $0, 1$ are fixed points and $\omega_\varphi(x) = \{0\}$ for any $x \in (0,1)$.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ \mathbb{F}(X) & \xrightarrow{\Phi} & \mathbb{F}(X) \end{array}$$

Our task - What are relations between dynamical properties of (X, φ) and $(\mathbb{F}(X), \Phi)$?

- [Kloeden, 1982] introduction to fuzzy discrete dynamical systems
- [Kloeden, 1991] sufficient conditions for Li-Yorke chaotic fuzzy dynamical systems
- [Diamond, Pokrovskii, 1994] chaos and entropy, relations to erratic maps
- [Bassanezi, de Barros, Tonelli, 2001] stability and attractors on the space of *n*-dimensional real fuzzy numbers
- [Pederson, 2005] homoclinic orbits of commuting fuzzifications
- [Roman-Flores, Chalco-Cano, 2008] transitivity, periodic density, sensitive dependence on initial conditions
- [Canovas, Kupka, 2011] topological entropy

Semiconjugacies:

$$
\begin{array}{ccc}
X & \stackrel{\varphi}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
\mathbb{K}(X) & \stackrel{\bar{\varphi}}{\longrightarrow} & \mathbb{K}(X) \\
\downarrow & & \downarrow \\
\mathbb{F}(X) & \stackrel{\Phi}{\longrightarrow} & \mathbb{F}(X)
\end{array}
$$

Outline

Notation: A ... a fuzzy set on a compact space X ($A : X \to I$)

- $[A]_\alpha$ ... an α-level set (α-cut) of A
- $\mathbb{F}(X)$ ... the family of upper semicontinuous fuzzy sets
  $\mathbb{F}^1(X)$ ... the family of normal fuzzy sets on X
- A is a fuzzy number on X iff $[A]_\alpha$ is connected for any $\alpha \in (0, 1)$
  $\mathbb{F}^1_c(X)$ ... the family of fuzzy numbers on X
- $\mathbb{K}(X)$ ... the metric space of nonempty compact subsets of X.
  $\mathbb{K}_c(X)$ ... the metric space of nonempty compact connected subsets of X.

Topological structures can be defined on these spaces.

For a given dynamical system (X, φ), we define its **fuzzification** or **Zadeh's extension** $\Phi : \mathbb{F}(X) \to \mathbb{F}(X)$ by

$$\Phi(A)(y) = \sup_{x \in \varphi^{-1}(y)} \{A(x)\}.$$

Properties:

- Φ is continuous
- Then, for any $\alpha \in (0, 1]$,

$$\varphi([A]_\alpha) = [\Phi(A)]_\alpha.$$

It is well-known that the dynamical system (X, φ) induces a dynamical system $(\mathbb{K}(X), \bar{\varphi})$ where

$$\bar{\varphi}(A) = \varphi(A) \text{ for any } A \in \mathbb{K}(X).$$

We distinguish three discrete dynamical systems:

- the **original** (**crisp**) one - (X, φ)
- the **set-valued** (**induced**) one - $(\mathbb{K}(X), \bar{\varphi})$
- the **fuzzy** (**fuzzified**) one - $(\mathbb{F}(X), \Phi)$

(Semi-)conjugacy

(X, φ), (Y, ψ) dynamical systems.

- A homeomorphism $g : X \to Y$ is a **conjugacy** iff

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ g\downarrow & & g\downarrow \\ Y & \xrightarrow{\psi} & Y \end{array}$$

- g only surjective $\Rightarrow$ **semiconjugacy**
- conjugated systems are dynamically the same, semiconjugated systems are "almost" the same
- many dynamical properties (dynamical invariants) are preserved by (semi-)conjugacies

Theorem

For any (X, φ), *there exists a semiconjugacy*

$$h : \mathbb{F}(X) \to \mathbb{K}(X).$$

Corollary

For any (X, φ), *there exists a semiconjugacy*

$$h : \mathbb{F}_c(X) \to \mathbb{K}_c(X).$$

Semiconjugacies:

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X \\
\not\uparrow h & & \not\uparrow h \\
\mathbb{K}(X) & \xrightarrow{\bar{\varphi}} & \mathbb{K}(X) \\
\uparrow g & & \uparrow g \\
\mathbb{F}(X) & \xrightarrow{\Phi} & \mathbb{F}(X)
\end{array}
$$

Remark. (X, φ) is **not** a dynamical factor of its set-valued and fuzzy extensions. Many recent results can be obtained as a consequence of this result.

Remark. Similar diagram can be constructed for the space of fuzzy numbers.

Topological entropy [Bowen, 1971]

Take X and fix $\varepsilon > 0$ and $n \in \mathbb{N}$. We say that a set $E \subset X$ is $(n, \varepsilon, \varphi)$**–separated** (by the map φ) if for any $x, y \in E$, $x \neq y$, there is $k \in \{0, 1, ..., n-1\}$ such that $d(\varphi^k(x), \varphi^k(y)) > \varepsilon$. Denote by $s_n(\varepsilon, \varphi)$ the cardinality of any maximal

$(n, \varepsilon, \varphi)$–separated set in X and define

$$s(\varepsilon, \varphi) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, \varphi).$$

It is known that $s(\varepsilon, \varphi)$ increases when ε decreases. Now the **topological entropy** of φ is

$$h_d(f) = \lim_{\varepsilon \to 0} s(\varepsilon, X, f).$$

Topological entropy [Bowen, 1971]

Take X and fix $\varepsilon > 0$ and $n \in \mathbb{N}$. We say that a set $E \subset X$ is $(n, \varepsilon, \varphi)$**–separated** (by the map φ) if for any $x, y \in E$, $x \neq y$, there is $k \in \{0, 1, ..., n-1\}$ such that $d(\varphi^k(x), \varphi^k(y)) > \varepsilon$. Denote by $s_n(\varepsilon, \varphi)$ the cardinality of any maximal

$(n, \varepsilon, \varphi)$–separated set in X and define

$$s(\varepsilon, \varphi) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, \varphi).$$

It is known that $s(\varepsilon, \varphi)$ increases when ε decreases. Now the **topological entropy** of φ is

$$h_d(f) = \lim_{\varepsilon\to 0} s(\varepsilon, X, f).$$

Topological entropy [Bowen, 1971]

Take X and fix $\varepsilon > 0$ and $n \in \mathbb{N}$. We say that a set $E \subset X$ is $(n, \varepsilon, \varphi)$**–separated** (by the map φ) if for any $x, y \in E$, $x \neq y$, there is $k \in \{0, 1, ..., n-1\}$ such that $d(\varphi^k(x), \varphi^k(y)) > \varepsilon$. Denote by $s_n(\varepsilon, \varphi)$ the cardinality of any maximal

$(n, \varepsilon, \varphi)$–separated set in X and define

$$s(\varepsilon, \varphi) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, \varphi).$$

It is known that $s(\varepsilon, \varphi)$ increases when ε decreases. Now the **topological entropy** of φ is

$$h_d(f) = \lim_{\varepsilon \to 0} s(\varepsilon, X, f).$$

Basic properties.

- $h(\varphi) \in [0, \infty]$
- h is monotone, i.e., $Y \subseteq X$ φ-invariant implies

$$h(\varphi|_Y) \leq h(\varphi)$$

- topological entropy is invariant w.r.t. conjugacy

Examples.

- $\varphi : I \to I$ homeo (e.g., $\varphi(x) = x^2$) ... $h(\varphi) = 0$
- homeo on $\mathbb{S}^1 \times \mathbb{S}^1$ can have positive topological entropy

If the space X is not compact, we use the following definition ([Canovas, 2005]) of topological entropy

$$\mathrm{ent}(\varphi) = \sup\{h(\varphi|_K) : (K, \varphi|_K) \text{ is a subsystem of } (X, \varphi)\},$$

i.e. K is a compact φ-**invariant** ($\varphi(K) \subseteq K$) subset of X.

Theorem

[with J. Canovas] *There exists a dynamical system* (X, φ) *possessing an trajectory of some point* $x \in X$ *containing infinite backward orbit. Then*

- $0 < h(\bar{\varphi}) < \infty$ *and*
- $ent_d(\Phi) = \infty$.

- Such assumptions can be easily satisfied.
- **Example.** Zadeh's extension of a homeomorphisms $h \neq id_{[0,1]}$ on $[0, 1]$ (e.g., $h(x) = x^2$) has infinite topological entropy

Theorem

[with J. Canovas] *There exists a dynamical system* (X, φ) *possessing an trajectory of some point* $x \in X$ *containing infinite backward orbit. Then*

- $0 < h(\bar{\varphi}) < \infty$ *and*
- $ent_d(\Phi) = \infty$.

- Such assumptions can be easily satisfied.
- **Example.** Zadeh's extension of a homeomorphisms $h \neq id_{[0,1]}$ on $[0, 1]$ (e.g., $h(x) = x^2$) has infinite topological entropy

- We showed that simple dynamics on the crisp maps produces very complicated dynamics of the Zadeh's extension.
- This result shows that set-valued and fuzzy extensions of (X, φ) can differ in sizes of their topological entropies.
- However, we dealt with numerous α-cuts.

- We showed that simple dynamics on the crisp maps produces very complicated dynamics of the Zadeh's extension.
- This result shows that set-valued and fuzzy extensions of (X, φ) can differ in sizes of their topological entropies.
- However, we dealt with numerous α-cuts.

- we studied topological entropy on $\mathbb{F}^1_c(X)$
- our results are studied for $X = I$
- it does not make sense to consider "smaller" spaces

Theorem

[with J. Canovas] *For* (I, φ). *Then*

$$h(\varphi) = ent(\Phi|_{\mathbb{F}^1_c(I)}).$$

- the same results for circles, graphs, trees, product maps etc.
- [Acosta et al., 2009]
  X ... a special dendrite, $\varphi : X \to X$ ... homeomorphism
  $h(\varphi) = 0$ while $ent(\Phi) = \infty$ on C where $C \subseteq \mathbb{F}^1_c(X)$

- due to proved (semi-)conjugacy many calculations can be done directly in the set-valued dynamical system
- the topological entropy is a suitable instrument to express the complexity of dynamics on the space of fuzzy numbers
- reasonable dynamical properties were specified for the space of fuzzy numbers on I
- our results and constructions can be extended to other spaces (graphs, circles, trees etc.) or to higher dimensions (by product or skew product maps ...)

Thank you for your attention.