Specificity measures and cardinalities of interval-valued fuzzy sets

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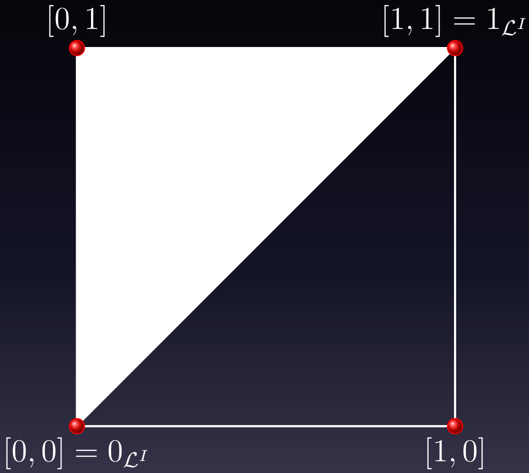
Interval-valued fuzzy set (IVFS)

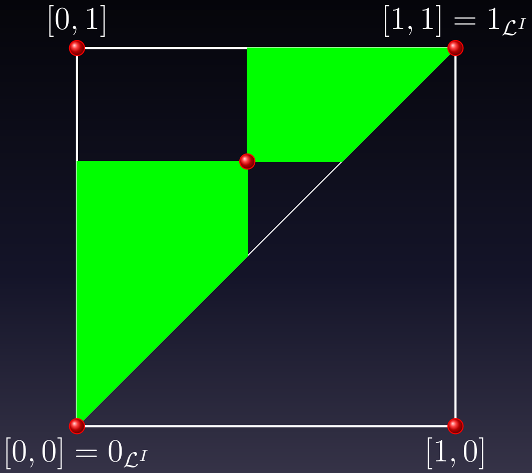
We define $\mathcal{L}^I = (L^I, \leq_{L^I})$ by

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2),$$

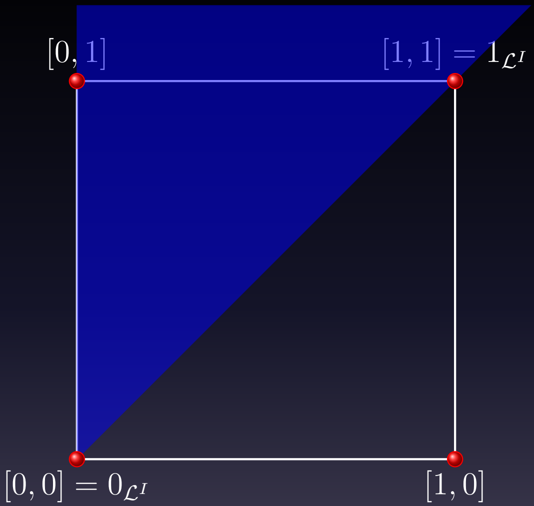
for all $[x_1, x_2], [y_1, y_2] \in L^I$.

An interval-valued fuzzy set on U is a mapping $A : U \to L^I$.


$[0,1]$
$[1,1] = 1_{\mathcal{L}^I}$
$[0,0] = 0_{\mathcal{L}^I}$
$[1,0]$



$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2)$$



$$\bar{L}^I_+ = \{[x_1, x_2] \mid (x_1, x_2) \in [0, +\infty[^2 \text{ and } x_1 \leq x_2\}$$

$$\bar{L}^I = \{[x_1, x_2] \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } x_1 \leq x_2\},$$
$$\bar{L}^I_+ = \{[x_1, x_2] \mid (x_1, x_2) \in [0, +\infty[^2 \text{ and } x_1 \leq x_2\},$$

Operations on $\bar{L}^I$, $\bar{L}^I_+$ and $\bar{L}^I_{+,0}$:

For all $a, b \in \bar{L}^I$, $c, d \in \bar{L}^I_+$ and $e, f \in \bar{L}^I_{+,0}$

$$a + b = [a_1 + b_1, a_2 + b_2],$$
$$c \cdot d = [c_1 d_1, c_2 d_2],$$
$$\frac{e}{f} = \left[\frac{e_1}{f_2}, \frac{e_2}{f_1}\right],$$

T-norms and related operations on $\mathcal{L}^I$ (Deschrijver)

- A t-norm on $\mathcal{L}^I$ is a commutative, associative mapping $\mathcal{T} : (L^I)^2 \rightarrow L^I$ which is increasing in both arguments and which satisfies $\mathcal{T}(1_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.
- A t-conorm on $\mathcal{L}^I$ is a commutative, associative mapping $\mathcal{S} : (L^I)^2 \rightarrow L^I$ which is increasing in both arguments and which satisfies $\mathcal{S}(0_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.
- A negation on $\mathcal{L}^I$ is a decreasing mapping $\mathcal{N} : L^I \rightarrow L^I$ which satisfies $\mathcal{N}(0_{\mathcal{L}^I}) = 1_{\mathcal{L}^I}$ and $\mathcal{N}(1_{\mathcal{L}^I}) = 0_{\mathcal{L}^I}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^I$, then $\mathcal{N}$ is called involutive.

Intersection, union, complement and inclusion

The generalized intersection $\cap_{\mathcal{T}}$, union $\cup_{\mathcal{S}}$ and complement $\mathrm{co}_{\mathcal{N}}$ of interval-valued fuzzy sets are defined as follows: for all $A, B \in \mathcal{F}_{\mathcal{L}^I}(U)$ and for all $u \in U$,

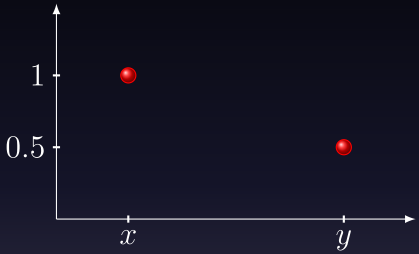
$$
\begin{aligned}
A \cap_{\mathcal{T}} B(u) &= \mathcal{T}(A(u), B(u)), \\
A \cup_{\mathcal{S}} B(u) &= \mathcal{S}(A(u), B(u)), \\
\mathrm{co}_{\mathcal{N}} A(u) &= \mathcal{N}(A(u)), \\
A \subseteq_{L^I} B &\iff A(u) \leq_{L^I} B(u).
\end{aligned}
$$

Scalar cardinalities

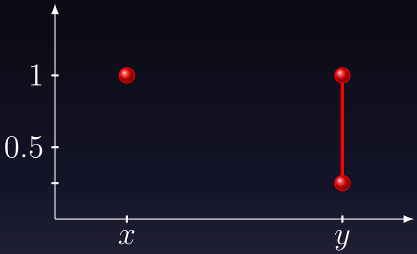
Human counting procedures in vague collections (Wygralak 2003, 2007, 2010)

- Wygralak, M.: Cardinalities of Fuzzy Sets, Springer, 2003
  - a more or less subjective process which may lead to different numerical results,
  - one needs to decide which elements are counted and what are their degrees of participation,
  - using a single threshold $\rightarrow$ scalar cardinalities,

$A = 1/x + 0.5/y$



$B = 1/x + [0.25, 1]/y$



$\mathrm{card}(A) = 1$
$\mathrm{card}(A) = 1.5$
$\mathrm{card}(A) = 2$

$\mathrm{card}(B) = [1, 1.5]$
$\mathrm{card}(B) = [1, 2]$

Deschrijver G., Kráľ, P.: On the cardinalities of interval-valued fuzzy sets, FSS 158 (2007)

Scalar cardinalities of IVFS

A mapping $\mathrm{card}_I : \mathcal{F}^F_{\mathcal{L}^I}(U) \to \bar{L}^I_+$ is called a scalar cardinality of interval-valued fuzzy sets if the following conditions hold:

1. coincidence: for all $u \in U$,

$$\mathrm{card}_I(1_{\mathcal{L}^I}/u) = 1_{\mathcal{L}^I};$$

2. monotonicity: for all $a, b \in L^I$ and $u, v \in U$,

$$a \leq_{L^I} b \implies \mathrm{card}_I(a/u) \leq_{L^I} \mathrm{card}_I(b/v);$$

3. additivity: for all $A, B \in \mathcal{F}^F_{\mathcal{L}^I}(U)$,

$$\mathrm{supp}(A) \cap \mathrm{supp}(B) = \emptyset \implies$$
$$\implies \mathrm{card}_I(A \cup B) = \mathrm{card}_I(A) + \mathrm{card}_I(B).$$

Representation theorem

A mapping $\mathrm{card}_I : \mathcal{F}^F_{\mathcal{L}^I}(U) \to \bar{L}^I_+$ is a scalar cardinality iff there exists a mapping $f_I : L^I \to L^I$ (called scalar cardinality pattern) fulfilling the following conditions:

1. $f_I(0_{\mathcal{L}^I}) = 0_{\mathcal{L}^I}$, $f_I(1_{\mathcal{L}^I}) = 1_{\mathcal{L}^I}$,
2. $f_I(a) \leq_{L^I} f_I(b)$ whenever $a \leq_{L^I} b$,

such that

$$\mathrm{card}_I(A) = \sum_{u \in \mathrm{supp}(A)} f_I(A(u)),$$

for each $A \in \mathcal{F}^F_{\mathcal{L}^I}(U)$.

1-semirepresentable cardinality patterns

1. Let $t \in D \setminus \{0_{\mathcal{L}^I}\}$ and f be a cardinality pattern of fuzzy sets. Define

$$f_{I,1_t}(a) = \begin{cases} [f(a_1), f(a_2)], & \text{if } a \geq_{L^I} t, \\ [0,0], & \text{otherwise,} \end{cases}$$

   for all $a \in L^I$.

2. Let $p \in \,]0,1]$, $\lambda \in [0,1[$. Define

$$f_{I,1_{p,\lambda}}(a) = \begin{cases} [1,1], & \text{if } a = 1_{\mathcal{L}^I}, \\ [0,0], & \text{if } a = 0_{\mathcal{L}^I}, \\ [a_1^p, \max\{a_1^p, \lambda\}], & \text{otherwise,} \end{cases}$$

   for all $a \in L^I$.

2-semirepresentable cardinality patterns

1. Let $t \in D \setminus \{1_{\mathcal{L}^I}\}$ and f be a cardinality pattern of fuzzy sets. Define

$$f_{I,2_t}(a) = \begin{cases} [0,0], & \text{if } a \leq_{L^I} t, \\ [f(a_1), f(a_2)], & \text{otherwise}, \end{cases}$$

   for all $a \in L^I$.

2. Let $p \in \,]0,1]$, $\lambda \in [0,1[$. Define

$$f_{I,2_{p,\lambda}}(a) = \begin{cases} [1,1], & \text{if } a = 1_{\mathcal{L}^I}, \\ [0,0], & \text{if } a = 0_{\mathcal{L}^I}, \\ [\min\{a_2^p, \lambda\}, a_2^p], & \text{otherwise}, \end{cases}$$

   for all $a \in L^I$.

Non-representable cardinality patterns

1. The largest cardinality pattern ${f_I}^{**}$:

$$ {f_I}^{**}(a) = \begin{cases} [1,1], & \text{if } a \neq 0_{\mathcal{L}^I}, \\ [0,0], & \text{otherwise,} \end{cases} $$

   for all $a \in L^I$.
2. The smallest cardinality pattern $f_{I_{**}}$:

$$ f_{I_{**}}(a) = \begin{cases} [1,1], & \text{if } a = 1_{\mathcal{L}^I}, \\ [0,0], & \text{otherwise,} \end{cases} $$

   for all $a \in L^I$.
3. Define

$$ f_{I_{\frac{1}{2}}}(a) = [0.5, 0.5], \text{ where } a \in L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}. $$

Specificity, non-specificity

Measures of the amount of information contained in a fuzzy subset

Hartley measure (1928)

$$\mathrm{H}(A) = \log_2(|A|)$$

- decision making, performance of knowledge-based systems
  - a measure of the tranquility of making a decision – the more specific the set of choices, the less anxiety provoking the decision (Yager, 1982),
  - specificity-correctness trade-off (Yager, 1982, 1984)
- approximate reasoning
  - minimal specificity principle (Dubois, Prade, 1987, 1995)

Specificity for FS (Yager, 1981, 1982)

Let U be a universe. A specificity for fuzzy sets is a mapping $\mathrm{SP} : \mathcal{F}^F_{[0,1]}(U) \to [0,1]$ such that, for all A, $B \in \mathcal{F}^F_{[0,1]}(U)$:

(S1) $\mathrm{SP}(A) = 1$ if A is a singleton,

(S2) $\mathrm{SP}(A) = 0$ if A is an empty set,

(S3) For normal fuzzy sets A, B it holds: $A \subseteq B$ then $\mathrm{SP}(A) \geq \mathrm{SP}(B)$.

$$\mathrm{Sp}(A) = \sum_{i=1}^{n} \frac{\mu_A(x_i) - \mu_A(x_{i+1})}{i}$$

Non-specificity for FS (Higashi, Klir, 1983)

Let U be a universe. A non-specificity for fuzzy sets is a mapping $\mathrm{NS} : \mathcal{F}^F_{[0,1]}(U) \to [0, +\infty]$ such that, for all A, $B \in \mathcal{F}^F_{[0,1]}(U)$:

(NS1) $\mathrm{NS}(A) = 1$ if A is a singleton,

(NS2) $A \subseteq B$ then $\mathrm{NS}(A) \leq \mathrm{NS}(B)$.

$$\mathrm{NS}(A) = \sum_{i=1}^{n} (\mu_A(x_i) - \mu_A(x_{i+1})) \log_2 i$$

Interval-valued non-specificity for IVFS

Let U be a universe. A non-specificity for fuzzy sets is a mapping $\mathrm{NSI} : \mathcal{F}^F_{[0,1]}(U) \to \bar{L}^I_+$ such that, for all A, $B \in \mathcal{F}^F_{[0,1]}(U)$:

(NSI1) $\mathrm{NSI}(A) = 1$ if A is a singleton,

(NSI2) $A \subseteq_{L^I} B$ then $\mathrm{NSI}(A) \leq_{L^I} \mathrm{NSI}(B)$.

Scalar cardinalities are non-specificities for IVFS.

Specificity for IVFS

(González-del-Campo, Garmendia, 2009)

Normal IVFS

An interval-valued fuzzy set A on a universe U is normal if there exists an element $u \in U$ such that $A(u) = 1_{\mathcal{L}^I}$.

Weak specificity

Let U be a universe. A weak specificity for interval-valued fuzzy sets is a mapping $\mathrm{SI} : \mathcal{F}^F_{\mathcal{L}^I}(U) \to [0, 1]$ such that, for all A, $B \in \mathcal{F}^F_{\mathcal{L}^I}(U)$:

(SI1) $\mathrm{SI}(A) = 1$ if A is a singleton,

(SI2) $\mathrm{SI}(A) = 0$ if A is an empty set,

(SI3) If A, B are normal interval-valued fuzzy sets such that $A \subseteq_{L^I} B$, then $\mathrm{SI}(A) \geq \mathrm{SI}(B)$.

Specificity for IVFS

(González-del-Campo, Garmendia, Yager, 2010)

Transformation operator

An operator $f : [0,1]^2 \to [0,1]$ with $x \leq y$ is called transformation operator if it is continuous, increasing and verifies

1. $f(1,1) = 1$,
2. $f(0,0) = 0$,
3. $f(0,x) > 0$ for all $x \in (0,1]$,
4. $f(x,1) < 1$ for all $x \in [0,1)$.

Example

$f(x,y) = \frac{x+y}{2}$, $f(x,y) = \frac{x^2+y^2}{2}$, $f(x,y) = \alpha \cdot x + \beta \cdot y$, where $\alpha + \beta = 1$, $\alpha > 0, \beta > 0$.

f-list

Let U be a universe. Let A be an interval-valued fuzzy set on U and let $\{[x_{1_q}, x_{2_q}]\}$, where $q \in \{1, \ldots, n\}$, be its membership intervals. Let f be a transformation operator. An f-list of A is the set of all membership intervals of elements belonging to $\mathrm{supp}(A)$ ordered decreasingly with respect to the operator f, i.e

$[x_1, x_2] \leq_f [y_1, y_2] \iff f(x_1, x_2) \leq f(y_1, y_2)$.

f-specificity

Let U be a universe. Let f be a transformation operator. Let $\{[x_{1_q}, x_{2_q}]\}$, where $q \in \{1, \ldots, n\}$, be an f-list of A. An f-specificity for interval-valued fuzzy sets is a mapping $\mathrm{SI_f} : \mathcal{F}^F_{\mathcal{L}^I}(U) \to [0,1]$ such that, for all A, $B \in \mathcal{F}^F_{\mathcal{L}^I}(U)$:

(SIF1) $\mathrm{SI_f}(A) = 1$ if A is a singleton,

(SIF2) $\mathrm{SI_f}(A) = 0$ if A is an empty set,

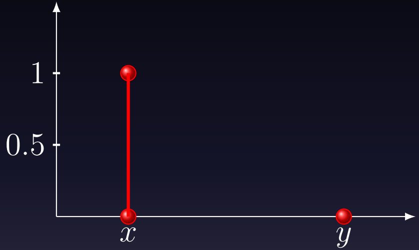
(SIF3) If $[x_{1_1}, x_{2_1}]$ increases (according to $\leq_{L^I}$) then $\mathrm{SI_f}(A)$ increases,

(SIF4) If $[x_{1_q}, x_{2_q}]$ increases (according to $\leq_{L^I}$), for all $q \in \{2, \ldots, n\}$, then $\mathrm{SI_f}(A)$ decreases.

$$\mathrm{SI_f}(A) = Q_F(a_i) - \frac{1}{n-1} \sum_{\forall k \neq i} Q_F(a_k),$$

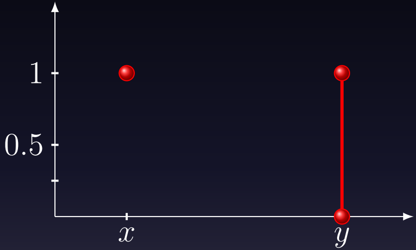
where $a_i = [x_{1_i}, x_{2_i}]$ maximizes Q_F and $x_1 \leq Q_F \leq x_2$

$$A = [0,1]/x + [0,0]/y$$



$\mathrm{SI}(A) = 1$
$\mathrm{SI}(A) = 0$
$\mathrm{SI}(A) \in [0,1]$

$$B = [1,1]/x + [0,1]/y$$



$\mathrm{SI}(B) = 1$
$\mathrm{SI}(B) = 0$
$\mathrm{SI}(B) \in [0.5,1]$

Interval-valued specificity for IVFS

Let U be a universe. An interval-valued specificity for interval-valued fuzzy sets is a mapping $\mathrm{SI} : \mathcal{F}^F_{\mathcal{L}^I}(U) \to L^U$ such that, for all A, $B \in \mathcal{F}^F_{\mathcal{L}^I}(U)$:

(SI1) $\mathrm{SI}(A) = 1_{\mathcal{L}^I}$ if A is a singleton,

(SI2) $\mathrm{SI}(A) = 0_{\mathcal{L}^I}$ if A is an empty set,

(SI3) If $A_2 = B_2 = 1$ and interval-valued fuzzy sets $A \subseteq_{L^I} B$, then $\mathrm{SI}(A) \geq_{L^I} \mathrm{SI}(B)$.

Non-representable specificities

1. The most restrictive specificity SI_*:

$$\mathrm{SI}_*(A) = \begin{cases} 1_{\mathcal{L}^I}, & \text{if } A \text{ is a singleton,} \\ 0_{\mathcal{L}^I}, & \text{otherwise,} \end{cases}$$

   for all $A \in \mathcal{F}^F_{\mathcal{L}^I}(U)$.

2. The least restrictive specificity

$$\mathrm{SI}^*(A) = \begin{cases} 0_{\mathcal{L}^I}, & \text{if } A \text{ is an empty set,} \\ 1_{\mathcal{L}^I}, & \text{otherwise,} \end{cases}$$

   for all $A \in \mathcal{F}^F_{\mathcal{L}^I}(U)$.

$$\mathrm{SI}_C(A) = \begin{cases} \dfrac{1_{\mathcal{L}^I}}{\mathrm{card}(A)}, & \text{if } A_1 > 0, \\ 0_{\mathcal{L}^I}, & \text{otherwise.} \end{cases}$$