NON-STANDARD MÖBIUS TRANSFORM-BASED EXTENSIONS OF BOOLEAN UTILITY FUNCTIONS

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1 Introduction

In this contribution

- we recall some recently studied Möbius transform-based extensions of Boolean normed utility functions to utility functions, generalizing the Lovász and Owen extensions of nondecreasing pseudo-Boolean functions,

- we introduce extensions of Boolean normed utility functions based on the generalized Möbius transform,

- we present some results on the extensions based on the possibilistic Möbius transform and give several examples, including extremal extensions of this type.

Basic notions

One of the problems in decision making theory is the problem of extending Boolean utility functions into utility functions.

Let $\mathfrak{X} = \{a_1, \ldots, a_k\}$ be a set of alternatives and $N = \{1, \ldots, n\}$ a set of criteria. Consider that the alternatives are crisp, i.e., characterized by $\{0,1\}$-valued score vectors $\mathbf{x} \in \{0,1\}^n$, and suppose they are evaluated by Boolean utility functions.

A (normed) Boolean utility function is a nondecreasing pseudo-Boolean function $u\colon \{0,1\}^n \to [0,1]$ satisfying the properties $u(0,\ldots,0) = 0$ and $u(1,\ldots,1) = 1$.

$\mathcal{B}_n$ - the set of all (normed) Boolean utility functions

1 Introduction

Boolean utility functions are in a one-to-one correspondence with fuzzy measures on the set N

A fuzzy measure (capacity) m on the set N is a nondecreasing set function $m\colon 2^N \to [0,1]$ with the properties $m(\emptyset) = 0$, $m(N) = 1$.

$$u\colon \{0,1\}^n \to [0,1] \;\leftrightarrow\; m_u\colon 2^N \to [0,1], \qquad m_u(E) = u(\mathbf{1}_E)$$

1 Introduction

If criteria are evaluated in the graded scale $[0,1]$, there is a need to extend Boolean utility functions to utility functions, i.e., for a given a Boolean utility function u we are looking for utility functions U such that $U|_{\{0,1\}^n} = u$.

A (normed) utility function is a nondecreasing function $U\colon [0,1]^n \to [0,1]$ which preserves the bounds $U(0,\ldots,0) = 0$, $U(1,\ldots,1) = 1$.

Normed utility functions $U\colon [0,1]^n \to [0,1]$ can be identified with n-ary aggregation functions on the interval $[0,1]$.

$\mathcal{A}_n$ - the set of all n-ary aggregation functions on $[0,1]$

1 Introduction

Recall two well-known extensions of Boolean utility functions u:

- Lovász's extension $U_{\mathcal{L}}$

$$U_{\mathcal{L}}(x_1, \ldots, x_n) = \sum_{E \subseteq N} M_{m_u}(E) \min_{i \in E} x_i,$$

- Owen's extension $U_{\mathcal{O}}$

$$U_{\mathcal{O}}(x_1, \ldots, x_n) = \sum_{E \subseteq N} M_{m_u}(E) \prod_{i \in E} x_i,$$

where $M_{m_u} : 2^N \to \mathbb{R}$ is the Möbius transform of the fuzzy measure m_u induced by u, given by

$$M_{m_u}(E) = \sum_{K \subseteq E} (-1)^{|E \setminus K|} m_u(K).$$

It holds that $U_{\mathcal{L}}$ is the Choquet integral of $\mathbf{x}\colon N \to [0,1]$, $\mathbf{x}(i) = x_i$, with respect to m_u [Chateauneuf and Jaffray (1989), Marichal (2002)]

2 Möbius transform-based extensions

The minimum and product are aggregation functions. Inspired by the Lovász and Owen extensions, for an n-ary aggregation function A and a Boolean utility function $u \in \mathcal{B}_n$ we can define the function $U_{u,A} \colon [0,1]^n \to \mathbb{R}$,

$$U_{u,A}(\mathbf{x}) = \sum_{E \subseteq N} M_{m_u}(E) A(\mathbf{x}_E)$$

where $\mathbf{x}_E = (v_1, \ldots, v_n)$, and

$$v_i = \begin{cases} x_i & \text{if } i \in E, \\ 1 & \text{otherwise.} \end{cases}$$

It holds

$$U_{\mathcal{L}} = U_{u,Min}, \qquad U_{\mathcal{O}} = U_{u,\Pi}$$

However, in general, the function $U_{u,A}$ is neither monotone nor an extension of u.

We have completely characterized all n-ary aggregation functions A leading for each Boolean utility function $u \in \mathcal{B}_n$ to a monotone extension of u:

Theorem

For an n-ary aggregation function A the following are equivalent.

(i) *For any Boolean utility function $u \in \mathcal{B}_n$, the function $U_{u,A}$ is a utility function extending u.*

(ii) *A is an aggregation function with zero annihilator and for any $\mathbf{a}, \mathbf{b} \in [0,1]^n$, $\mathbf{a} \leq \mathbf{b}$, such that*

$$\{0,1\} \cap \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \neq \emptyset,$$

the A-volume $V_A([\mathbf{a}, \mathbf{b}])$ is non-negative.

Aggregation functions characterized by the theorem are called **suitable aggregation functions**

Corollary

Let C be an n-copula, $\varphi_i\colon [0,1] \to [0,1]$, $i = 1, \dots, n$, nondecreasing functions such that for each i, $\varphi_i(0) = 0$, $\varphi_i(1) = 1$. Then the function $A\colon [0,1]^n \to [0,1]$ defined by

$$A(x_1, \dots, x_n) = C\left(\varphi_1(x_1), \dots, \varphi_n(x_n)\right)$$

(distorted copula) is a suitable n-ary aggregation function.

Characterization of suitable binary aggregation functions:

Theorem

Let $A \in \mathcal{A}_{(2)}$. For each $u \in \mathcal{B}_{(2)}$, the function $U_{u,A}$ is a utility function extending u if and only if for each $(x, y) \in [0,1]^2$ it holds

$$A(x, y) = Q(\varphi(x), \psi(y)),$$

where Q is a 2-quasi-copula and φ, ψ are nondecreasing $[0,1] \to [0,1]$ functions with $\varphi(0) = \psi(0) = 0$, $\varphi(1) = \psi(1) = 1$.

3 Non-standard Möbius transform-based extensions

We generalize the previous approach based on the standard operations $(+,\cdot)$ by means of so-called Pan-operations $(\oplus,\odot)$ on $[0,\infty]$, supposing that $e=1$ is the neutral element of $\odot$.

Pan-additions:

■ Archimedean pseudo-additions (continuous Archimedean t-conorms on $[0,\infty]$), which can be expressed in the form

$$x \oplus y = g^{-1}\left(g(x)+g(y)\right), \tag{1}$$

where $g\colon [0,\infty] \to [0,\infty]$ is an automorphism

■ $\oplus = \vee$.

Pan additions can be extended to $[-\infty,\infty]$. If $\oplus$ is generated by g, the extension of $\oplus$ is defined via (1) by means of the odd extension of g (with convention $-\infty+\infty=\infty$).

Pan-multiplications:

- For $\oplus = \oplus_g$ there is the only Pan-multiplication with neutral element $e = 1$, given by

$$x \odot_g y = g^{-1}\left(\frac{g(x) \cdot g(y)}{g(1)}\right).$$

The constant $g(1)$ has no influence, we will always suppose that $g(1) = 1$, hence

$$x \odot_g y = g^{-1}\left(g(x) \cdot g(y)\right). \tag{2}$$

Again, Pan-multiplications can be extended to $[-\infty, \infty]$.

- If $\oplus = \vee$, as an operation $\odot$ we can take any aggregation function on $[0, \infty]$ with neutral element $e = 1$ ($\odot|_{[0,1]^2}$ is a semicopula).

Pseudo-Möbius transform:

- The generalized Möbius transform of a fuzzy measure m, based on a Pan-addition $\oplus = \oplus_g$ is given by

$$M_m{}^{\oplus_g}(E) = \bigoplus_{K \subseteq E} (-1)^{|E \setminus K|} m(K) = g^{-1}\left(M_{g \circ m}(E)\right), \quad E \subseteq N, \tag{3}$$

[Mesiar (1997)]

- For $\oplus = \vee$, $M_m^{\vee}$ is defined by

$$M_m^{\vee}(E) = \begin{cases} m(E) & \text{if } \nexists F \subsetneq E,\ m(F) = m(E), \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

[Mesiar (1997); Marichal (1998)]

Now, for any Boolean utility function $u \in \mathcal{B}_n$ and any n-ary aggregation function $A \in \mathcal{A}_n$ we can define the function

$U_{u,A}^{\oplus,\odot} \colon [0,1]^n \longrightarrow \mathbb{R}$ by

$$U_{u,A}^{\oplus,\odot}(\mathbf{x}) = \bigoplus_{E \subseteq N} M_{m_u}^{\oplus}(E) \odot A(\mathbf{x}_E).$$

For generated operations $\oplus_g$ and $\odot_g$ it holds

$$U_{u,A}^{\oplus_g,\odot_g}(\mathbf{x}) = g^{-1}\left(\sum_{E\subseteq N} M_{g\circ m_u}(E)g(A(\mathbf{x}_E))\right)$$

and next

$$U_{u,A}^{\oplus_g,\odot_g}(\mathbf{x}) = g^{-1}\left(U_{g\circ u, g\circ A}(\mathbf{x})\right).$$

Characterization of $(\oplus_g, \odot_g)$-suitable aggregation functions:

Proposition

For each $u \in \mathcal{B}_n$, the function $U_{u,A}^{\oplus_g,\odot_g}$ is a utility function (aggregation function) if and only if $g \circ A$ is a suitable aggregation function, i.e., if and only if 0 is the annihilator of A and $V_{g\circ A}([\mathbf{a}, \mathbf{b}]) \geq 0$ for each n-box $[\mathbf{a}, \mathbf{b}]$ in $[0, 1]^n$ such that $\{a_1, \ldots, a_n, b_1, \ldots, b_n\} \cap \{0, 1\} \neq \emptyset$.

Example

Let $g(x) = x^p$, $p > 0$. *Then*

$$x \oplus_p y = (x^p + y^p)^{1/p}, \quad x \odot_p y = xy.$$

Let $u \in \mathcal{B}_2$ *be given by* $u(1,0) = a$, $u(0,1) = b$, $a, b \in [0,1]$ *(i.e.* $m_u(\{1\}) = a$, $m_u(\{2\}) = b$*). For an aggregation function* $A \in \mathcal{A}_2$ *we have*

$$U_{u,A}^{\oplus_p,\odot_p}(x,y) = (a^p A(x,1)^p + b^p A(1,y)^p + (1 - a^p - b^p) A(x,y)^p)^{1/p}.$$

Especially, if $a^p + b^p = 1$, *we have*

$$U_{u,A}^{\oplus_p,\odot_p}(x,y) = (a^p A(x,1)^p + (1 - a^p) A(1,y)^p)^{1/p}.$$

Example (Continued)

Using the characterization of suitable binary aggregation functions in the classical case, we get that $A \in \mathcal{A}_2$ *is* $(\oplus_g, \odot_g)$*-suitable if and only if there is a binary quasi-copula* Q *and increasing functions* φ, $\varphi \colon [0,1] \to [0,1]$, *with properties* $\varphi(0) = \psi(0) = 0$, $\varphi(1) = \psi(1) = 1$ *such that*

$$g \circ A(x, y) = Q(\varphi(x), \psi(y)).$$

For $g(x) = x^p$ *this can be written as*

$$A(x, y) = \left(Q\left(\varphi(x), \psi(y)\right)\right)^{1/p}.$$

Example (Continued)

- *Clearly, the case $p=1$, $Q = Min$ and $\varphi = \psi = id_{[0,1]}$, gives the classical result*

$$U_{u,Min}^{\oplus_p,\odot_p}(\mathbf{x}) = U_{\mathcal{L}}(\mathbf{x}) = C - \int_N \mathbf{x}\mathrm{d}m_u.$$

- *In general, for $g(x) = x^p$ the function $A(x,y) = (Min(x,y))^{1/p}$ leads to the utility function*

$$U_{u,A}^{\oplus_p,\odot_p}(\mathbf{x}) = \left(\sum_{E\subseteq N} M_{m_u^p}(E)Min(\mathbf{x}_E)\right)^{1/p} = \left(C - \int_N \mathbf{x}\mathrm{d}m_u^p\right)^{1/p}.$$

This is a new type of aggregation function different from the Choquet-like integral given by $\left(C - \int_N \mathbf{x}^p\mathrm{d}m^p\right)^{1/p}$ (introduced by Mesiar (1995)).

Consider a couple of Pan-operations $(\vee, \odot)$, where $\odot|_{[0,1]^2}$ is a semicopula. Recall that

$U_{u,A}^{\vee,\odot}\colon [0,1]^n \to [0,1]$ is given by

$$U_{u,A}^{\vee,\odot}(\mathbf{x}) = \bigvee_{E \subseteq N} \left(M_u^{\vee}(E) \odot A(\mathbf{x}_E) \right)$$

Theorem

Let A be an n-ary aggregation function. For each Boolean utility function $u \in \mathcal{B}_n$, the function $U_{u,A}^{\vee,\odot}$ is a utility function extending u if and only if A is an aggregation function with zero annihilator.

Remark

In the case of possibilistic Möbius transform the function $U^{\vee,\odot}_{u,A}$ *is nondecreasing for each fixed* A, u *and* $\odot$. *However, if zero is not the annihilator of* A *we can find a semicopula* $\odot$, *a Boolean utility function* $u \in \mathcal{B}_n$ *and* $\mathbf{x} \in \{0,1\}^n$ *such that* $U^{\vee,\odot}_{u,A}(\mathbf{x}) > u(\mathbf{x})$. *Moreover, in any case* $U^{\vee,\odot}_{u,A} \geq A$.

Example

Let $n = 2$. *Put* $u(1,0) = 0.2$, $u(0,1) = 0.6$. *Let* $\odot = \cdot$ *and* A *the arithmetic mean. Then*

$$U^{\vee,\cdot}_{u,A}(1,0) = (0.2)\cdot 1 \vee (0.6)\cdot(0.5) \vee 1\cdot(0.5) = 0.5 > 0.2 = u(1,0)\,.$$

Remark

(i) *The function* $U^{\vee,\odot}_{u,A}$ *can be written equivalently as*

$$U^{\vee,\odot}_{u,A}(\mathbf{x}) = \bigvee_{E\subseteq N} u(\mathbf{1}_E) \odot A(\mathbf{x}_E).$$

(ii) *Put* $\odot = \wedge$ *and* $A = Min$. *Then*

$$U^{\vee,\wedge}_{u,Min}(\mathbf{x}) = Su - \int_N \mathbf{x} \mathrm{d}m_u.$$

(Compare with the relation $U_{u,Min}(\mathbf{x}) = C - \int_N \mathbf{x} \mathrm{d}m_u$*).*

Example

Let $n = 2$, $u \in \mathcal{B}_2$, $A \in \mathcal{A}_2$. *Then*

$$U^{\vee,\odot}_{u,A}(x,y) = (u(1,0) \odot A(x,1)) \vee (u(0,1) \odot A(1,y)) \vee A(x,y).$$

If we denote $u(1,0) = a$, $u(0,1) = b$, *then for a semicopula* A *we get*

$$U^{\vee,\odot}_{u,A}(x,y) = (a \odot x) \vee (b \odot y) \vee A(x,y),$$

which is:

- the Sugeno integral if $\odot = \wedge$ *and* $A = Min$ *(Fig.1);*

- the Shilkret integral if $\odot = \cdot$ *and* $A = Min$ *(Fig.2).*

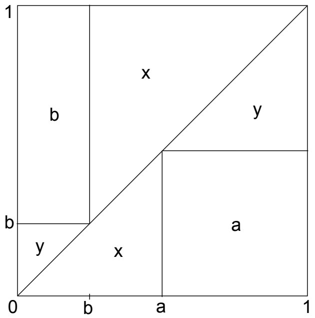

Fig.1: The extension $U_{u,Min}^{\vee,\wedge}$ (Sugeno integral)

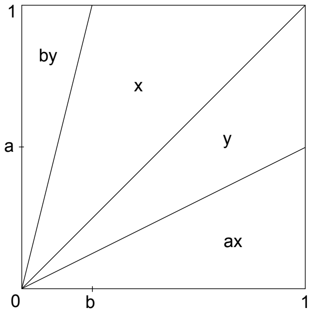

Fig.2: The extension $U_{u,Min}^{\vee,\cdot}$ (Shilkret integral)

Remark

For any semicopula $\odot$, the function $U^{\vee,\odot}_{u,Min}$ coincides with the weakest universal integral $\mathcal{I}_{\odot}$ based on $\odot$ with respect to the capacity m_u induced by u (Klement et al. (2010))

Example

Let A_* be the weakest n-ary aggregation function,

$$A_*(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $U^{\vee,\odot}_{u,A_*}$ does not depend on $\odot$ and it is the weakest extension of a Boolean utility function $u \in \mathcal{B}_n$ to a utility function. It holds:

$$U^{\vee,\odot}_{u,A_*}(\mathbf{x}) = m_u(Ker(\mathbf{x})),$$

where $Ker(\mathbf{x}) = \{i \in N \mid x_i = 1\}$.

Example

Let A^* be the strongest n-ary aggregation function with zero annihilator,

$$A^*(\mathbf{x}) = \begin{cases} 0 & \text{if } 0 \in \{x_1, \ldots, x_n\}, \\ 1 & \text{otherwise}, \end{cases}$$

Then $U^{\vee,\odot}_{u,A^*}$ does not depend on $\odot$ and it is the strongest extension of a Boolean utility function $u \in \mathcal{B}_n$ to a utility function. It holds

$$U^{\vee,\odot}_{u,A^*}(\mathbf{x}) = m_u\left(Supp(\mathbf{x})\right),$$

where $Supp(\mathbf{x}) = \{i \in N \mid x_i > 0\}$.

6 Concluding remarks

The main contribution of this presentation is the introduction and discussion of new types of extensions of Boolean normed utility functions based on the generalized Möbius transform, especially, on the possibilistic Möbius transform (with the supremum and a semicopula as the operations), which cover, among others, the concept of the weakest universal integrals. This approach can also be applied in the case when an ordinal scale for evaluating the satisfaction degrees of single criteria is considered.

THANK YOU FOR YOUR ATTENTION !