An axiomatic approach to some fuzzy integrals

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Contents

Introduction

Fixed finite space $X = \{1, \ldots, n\}$, functions from X to $[0, 1]$
we identify with vectors $\mathbf{x} = (x_1, \ldots, x_n)$
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1) construction based on capacity (measure)
$m : 2^X \to [0, 1], \quad U = I(m, \bullet)$
2) axiomatic approach

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Lebesgue integral

1) additive capacity = probability

$$m(A) = \sum_{i \in A} w_i, \quad I(m, \mathbf{x}) = U(\mathbf{x}) = \sum_{i=1}^{n} w_i \, x_i$$

2) U is additive

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A *capacity* on X $m\colon 2^X \to [0,1]$ which is non-decreasing, i.e., we have $m(E) \leq m(F)$ whenever $E \subseteq F \subseteq X$, $m(\emptyset) = 0$ and $m(X) = 1$.

Choquet integral

$$\mathbf{Ch}(m, \mathbf{x}) = \int_0^1 m(\{i \mid x_i \geq t\})\, dt =$$

$$= \sum_{i=1}^{n} x_{\pi_i} \cdot (m(\{\pi_i, \ldots, \pi_n\}) - m(\{\pi_{i+1}, \ldots, \pi_n\})),$$

for some permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of $\{1, \ldots, n\}$ satisfying $x_{\pi_1} \leq x_{\pi_2} \leq \cdots \leq x_{\pi_n}$

Sugeno integral

$$\mathbf{Su}(m, \mathbf{x}) = \bigvee_{t=0}^{1} (t \wedge m(\{i \mid x_i \geq t\}) = \bigvee_{i=1}^{n} (x_{\pi_i} \wedge m(\{\pi_i, \ldots, \pi_n\})).$$

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Let $\mathbf{x}, \mathbf{y} \in [0,1]^n$. Then $\mathbf{x}$ and $\mathbf{y}$ are said to be *comonotone* if, for all $i, j \in \{1, 2, \ldots, n\}$, we have $(x_i - x_j) \cdot (y_i - y_j) \geq 0$.

In other words, for comonotone $\mathbf{x}, \mathbf{y} \in [0,1]^n$ it is impossible to have $x_i > x_j$ and $y_i < y_j$.

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(i) An (*n-dimensional*) *aggregation function* is a function $A\colon [0,1]^n \to [0,1]$ which is non-decreasing in each component and satisfies the boundary conditions $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$.

(ii) An aggregation function $A\colon [0,1]^n \to [0,1]$ is said to be *comonotone additive* if, for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$ which are comonotone and satisfy $\mathbf{x} + \mathbf{y} \in [0,1]^n$, we have

$$U(\mathbf{x} + \mathbf{y}) = U(\mathbf{x}) + U(\mathbf{y}).$$

Proposition

(Schmeidler 1986) Let $U\colon [0,1]^n \to [0,1]$ *be an n-ary aggregation function. Then the following are equivalent:*

(i) *There is a capacity* $m\colon 2^X \to [0,1]$ *such that* $U(\cdot) = \mathbf{Ch}(m,\cdot)$.

(ii) *U is comonotone additive.*

Proposition

(Marichal 2001) Let $U\colon [0,1]^n \to [0,1]$ *be an n-ary aggregation function. Then the following are equivalent:*

(i) *There is a capacity* $m\colon 2^X \to [0,1]$ *such that* $U(\cdot) = \mathbf{Su}(m,\cdot)$.

(ii) U *is* $\wedge$*-homogeneous and comonotone maxitive, i.e., for each* $c \in [0,1]$, *the constant score vector* $\mathbf{c} = (c,\ldots,c)$ *and all comonotone* $\mathbf{x},\mathbf{y} \in [0,1]^n$ *we have*

$$U(\mathbf{c} \wedge \mathbf{x}) = c \wedge U(\mathbf{x}),$$
$$U(\mathbf{x} \vee \mathbf{y}) = U(\mathbf{x}) \vee U(\mathbf{y}).$$

A *semicopula* is two-dimensional aggregation function $\odot\colon [0,1]^2 \to [0,1]$ with neutral element 1.

Let $\odot\colon [0,1]^2 \to [0,1]$ be a semicopula and let $m\colon 2^X \to [0,1]$ be a capacity on X. A *discrete universal integral* (*based on* $\odot$) is an aggregation function $\mathbf{I}_{\odot,m}\colon [0,1]^n \to [0,1]$ such that

(i) for all $c \in [0,1]$ and all $E \subseteq X$ we have $\mathbf{I}_{\odot,m}(c \cdot \mathbf{1}_E) = c \odot m(E)$;

(ii) for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$ with $m(\{i \in X \mid x_i \geq t\}) = m(\{j \in X \mid y_j \geq t\})$ for all $t \in [0,1]$ we have $\mathbf{I}_{\odot,m}(\mathbf{x}) = \mathbf{I}_{\odot,m}(\mathbf{y})$.

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Observe that if a capacity m assumes values in $\{0, 1\}$ only then all discrete universal integrals are independent of the semicopula $\odot$, and they correspond to lattice polynomials.

A (*binary*) *copula* $C\colon [0,1]^2 \to [0,1]$ is a semicopula which is supermodular, i.e., for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y}).$$

Proposition

Let $C\colon [0,1]^2 \to [0,1]$ be a copula and $m\colon 2^X \to [0,1]$ a capacity, and define $\mathbf{K}_C(m,\cdot)\colon [0,1]^n \to [0,1]$ by

$$\mathbf{K}_C(m,\mathbf{x}) = \sum_{i=1}^{n} (C(x_{\pi_i}, m(\{\pi_i,\ldots,\pi_n\}) - C(x_{\pi_{i-1}}, m(\{\pi_i,\ldots,\pi_n\})),$$

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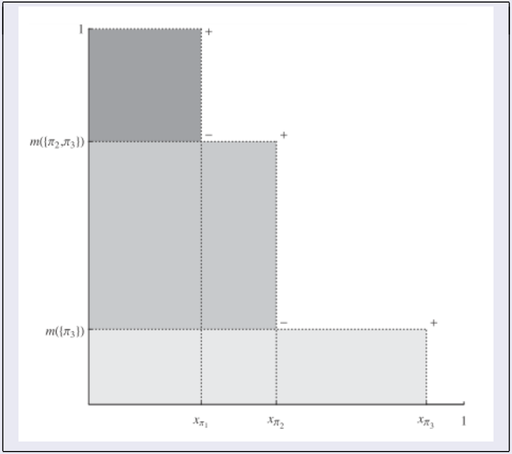

Figure: Copula-based universal integral K_C

$$\mathbf{K}_C(m,\mathbf{x}) = C(x_{\pi_1},1) - C(x_{\pi_1}, m(\{\pi_2,\pi_3\})) + C(x_{\pi_2}, m(\{\pi_2,\pi_3\})) - $$
$$- C(x_{\pi_2}, m(\{\pi_3\})) + C(x_{\pi_3}, m(\{\pi_3\}))$$

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for the product copula Π, $\Pi(x,y) = x \cdot y$

$$\mathbf{K}_\Pi(m,\mathbf{x}) = \sum_{i=1}^{n} x_{\pi_i} \cdot (m(\{\pi_i,\dots,\pi_n\}) - m(\{\pi_{i+1},\dots,\pi_n\}))$$

$\mathbf{K}_\Pi$ coincides with the Choquet integral

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is the smallest universal integral linked to $\odot$

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Proposition

Let C be a copula and m a capacity on X. Then $\mathbf{K}_C(m,\cdot)$ is a comonotone modular aggregation function, i.e., for all comonotone $\mathbf{x},\mathbf{y}\in[0,1]^n$

$$\mathbf{K}_C(m,\mathbf{x}\vee\mathbf{y})+\mathbf{K}_C(m,\mathbf{x}\wedge\mathbf{y})=\mathbf{K}_C(m,\mathbf{x})+\mathbf{K}_C(m,\mathbf{y}).$$

Define $U\colon[0,1]^2\to[0,1]$ by $U(x,y)=(x\wedge\frac{1}{2})+((y-\frac{1}{2})\vee 0)$. Then U is an idempotent modular, but there is no copula C so that

$$U=K_C(m,\bullet)$$

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$$U=K_C\,(m,\bullet)$$

Theorem

Let $U\colon [0,1]^n \to [0,1]$ be an aggregation function. Then the following are equivalent:

(i) *there is a copula C and a capacity m on X such that $U(\cdot) = \mathbf{K}_C(m, \cdot)$;*

(ii) *U is idempotent and comonotone modular, and for all $E, F \subseteq X$ and all $(u, v) \in [0,1]^2$ we have*

$$U(\mathbf{1}_E) = U(\mathbf{1}_F) \Rightarrow U(u \cdot \mathbf{1}_E) = U(u \cdot \mathbf{1}_F)$$

$$u \leq v \text{ and } U(\mathbf{1}_E) \leq U(\mathbf{1}_F) \Rightarrow$$
$$U(u \cdot \mathbf{1}_F) - U(u \cdot \mathbf{1}_E) \leq U(v \cdot \mathbf{1}_F) - U(v \cdot \mathbf{1}_E)$$

Symmetry of an aggregation function $U\colon [0,1]^n \to [0,1]$ means that we have $U(x_1,\dots,x_n) = U(x_{\pi_1},\dots,x_{\pi_n})$ for each permutation $(\pi_1,\dots,\pi_n)$. Symmetry of a capacity means that we have $m(E) = m(\{\pi_i \mid i \in E\})$ for each $E \subseteq X$ and for each permutation $(\pi_1,\dots,\pi_n)$, i.e., $m(E) = m(F)$ whenever $E, F \subseteq X$ have the same cardinality.

Theorem

Let $U\colon [0,1]^n \to [0,1]$ be a symmetric aggregation function. Then the following are equivalent:

(i) *there is a copula C and a symmetric capacity m on X such that $U(\cdot) = \mathbf{K}_C(m,\cdot)$;*

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(ii) *U is idempotent and comonotone modular.*

(i) 2011 Mesiar, Zemánková: *Ordered Modular Averages* (*OMA operators*)

(ii) 1988 Yager OWA operators, $\mathbf{K}_{\Pi}(m, \cdot)$ with respect to a symmetric capacity m is characterized up to symmetry by the comonotone additivity

(iii) $\mathbf{K}_{M}(m, \cdot)$ with respect to a symmetric capacity m is an *Ordered Weighted Maximum* (*OWMax operator*) - 1991 Dubois, Prade. It is characterized by symmetry, comonotone maxitivity and $\wedge$-homogeneity.

Theorem

Let $U\colon [0,1]^n \to [0,1]$ be an idempotent aggregation function. Then the following are equivalent:

(i) *there is a capacity $m\colon 2^X \to [0,1]$ and a semicopula $\odot\colon [0,1]^2 \to [0,1]$ such that $U = \mathbf{I}_{\odot,m}$;*

(ii) *U is comonotone maxitive and for all $E, F \subseteq X$ with $U(\mathbf{1}_E) \leq U(\mathbf{1}_F)$ and for each each $t \in]0,1[$ we have $U(t \cdot \mathbf{1}_E) \leq U(t \cdot \mathbf{1}_F)$.*

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(ii) U *is comonotone maxitive and for all* $E, F \subseteq X$ *with* $U(\mathbf{1}_E) = U(\mathbf{1}_F)$ *and for each each* $t \in]0,1[$ *we have* $U(t \cdot \mathbf{1}_E) = U(t \cdot \mathbf{1}_F)$.

Note that we cannot omit the condition that $U(\mathbf{1}_E) = U(\mathbf{1}_F)$ implies $U(t \cdot \mathbf{1}_E) = U(t \cdot \mathbf{1}_F)$. Define $U\colon [0,1]^2 \to [0,1]$ by $U(x,y) = \wedge(x,y) \vee (\vee(x,y))^2$. Then U is a symmetric, idempotent and comonotone maxitive aggregation function. If we define $m\colon 2^X \to [0,1]$ by $m(E) = U(\mathbf{1}_E)$ we see that $m(E) = 1$ whenever $E \neq \emptyset$. However, then for each semicopula $\odot\colon [0,1]^2 \to [0,1]$ we get $\mathbf{I}_{\odot,m} = \vee \neq U$.

Thanks for your attention