Aggregation functions and ultramodularity

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FSTA 2012, Liptovský Ján
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Let $I$ be a subinterval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a real function.

(i) $f$ is said to be **convex** if, for all $x, y \in I$ and for all $\lambda \in [0, 1]$,

$$\lambda \cdot f(x) + (1 - \lambda) \cdot f(y) \geq f(\lambda \cdot x + (1 - \lambda) \cdot y); \quad (1)$$

(ii) $f$ is said to be **Jensen convex** if, for all $x, y \in I$,

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x + y}{2}\right). \quad (2)$$
Let $I$ be a subinterval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a real function. Then we have:

(i) $f$ is convex if and only if, for all $x, y \in I$ and for all $\varepsilon > 0$ such that $x < y$ and $y + \varepsilon \in I$,

$$f(y + \varepsilon) - f(y) \geq f(x + \varepsilon) - f(x).$$

(ii) If $f$ is a continuous function then $f$ is convex if and only if it is Jensen convex.

(iii) If $f$ is a monotone function then $f$ is convex if and only if it is Jensen convex.

(iv) If $f$ is a bounded function then $f$ is convex if and only if it is Jensen convex.
Let \((L, \land, \lor)\) be a lattice.

(i) A function \(f: L \to \mathbb{R}\) is called **modular** if, for all \(x, y \in L\),

\[
f(x \lor y) + f(x \land y) = f(x) + f(y).
\]  

(ii) A function \(f: L \to \mathbb{R}\) is called **supermodular** if, for all \(x, y \in L\),

\[
f(x \lor y) + f(x \land y) \geq f(x) + f(y).
\]
Proposition

An n-ary aggregation function $A: [0, 1]^n \rightarrow [0, 1]$ is modular if and only if there are non-decreasing functions $f_1, f_2, \ldots, f_n: [0, 1] \rightarrow [0, 1]$ with

$$\sum_{i=1}^n f_i(0) = 0 \text{ and } \sum_{i=1}^n f_i(1) = 1$$

such that, for all $(x_1, \ldots, x_n) \in [0, 1]^n$,

$$A(x_1, \ldots, x_n) = \sum_{i=1}^n f_i(x_i).$$
For non-decreasing functions $f: [0, 1]^2 \rightarrow [0, 1]$, supermodularity can be reformulated as

$$f(x^*, y^*) - f(x^*, y) - f(x, y^*) + f(x, y) \geq 0$$  \hspace{1cm} (6)$$

for all $x, x^*, y, y^* \in [0, 1]$ with $x \leq x^*$ and $y \leq y^*$.

**Proposition**

An $n$-ary function $f: [0, 1]^n \rightarrow [0, 1]$ is supermodular if and only if each of its two-dimensional sections is supermodular, i.e., for each $x \in [0, 1]^n$ and all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, the function $f_{x, i, j}: [0, 1]^2 \rightarrow [0, 1]$ given by $f_{x, i, j}(u, v) = f(y)$, where $y_i = u$, $y_j = v$ and $y_k = x_k$ for $k \in \{1, 2, \ldots, n\} \setminus \{i, j\}$, is supermodular.
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The supermodularity of a function $f: [0, 1]^n \rightarrow [0, 1]$ is preserved if the arguments are distorted, i.e., if $g_1, \ldots, g_n: [0, 1] \rightarrow [0, 1]$ are non-decreasing functions, then the function $h: [0, 1]^n \rightarrow [0, 1]$ given by $h(\mathbf{x}) = f(g_1(x_1), \ldots, g_n(x_n))$ is supermodular (if $f$ is a supermodular aggregation function with $f(g_1(0), \ldots, g_n(0)) = 0$ and $f(g_1(1), \ldots, g_n(1)) = 1$ then $h$ is also a supermodular aggregation function).
Definition

An aggregation function $C : [0, 1]^2 \rightarrow [0, 1]$ is called a 2-\textit{copula} (or, briefly, a \textit{copula}) if it is supermodular and has 1 as neutral element, i.e., if $C(x, 1) = C(1, x) = x$ for all $x \in [0, 1]$. 
Proposition

An aggregation function $A: [0, 1]^2 \rightarrow [0, 1]$ is supermodular if and only if there are non-decreasing functions $g_1, g_2, g_3, g_4: [0, 1] \rightarrow [0, 1]$ with $g_i(0) = 0$ and $g_i(1) = 1$ for $i \in \{1, 2, 3, 4\}$, a copula $C: [0, 1]^2 \rightarrow [0, 1]$, and numbers $a, b, c \in [0, 1]$ with $a + b + c = 1$ such that, for all $(x, y) \in [0, 1]^2$,

$$A(x, y) = a \cdot g_1(x) + b \cdot g_2(y) + c \cdot C(g_3(x), g_4(y)).$$  (7)
Definition

An $n$-ary aggregation function $A: [0, 1]^n \rightarrow [0, 1]$ is called ultramodular if, for all $x, y, z \in [0, 1]^n$ with $x + y + z \in [0, 1]^n$,

$$A(x + y + z) - A(x + y) \geq A(x + z) - A(x).$$  \hspace{1cm} (8)

In the case of one-dimensional aggregation functions, ultramodularity is just standard convexity. Therefore, ultramodularity can also be seen as an extension of one-dimensional convexity.
**Definition**

An \( n \)-ary aggregation function \( A: [0, 1]^n \rightarrow [0, 1] \) is called *ultramodular* if, for all \( \mathbf{x}, \mathbf{y}, \mathbf{z} \in [0, 1]^n \) with \( \mathbf{x} + \mathbf{y} + \mathbf{z} \in [0, 1]^n \),

\[
A(\mathbf{x} + \mathbf{y} + \mathbf{z}) - A(\mathbf{x} + \mathbf{y}) \geq A(\mathbf{x} + \mathbf{z}) - A(\mathbf{x}). \tag{8}
\]

In the case of one-dimensional aggregation functions, ultramodularity is just standard convexity. Therefore, ultramodularity can also be seen as an extension of one-dimensional convexity.
Proposition

A function $f : [0, 1]^n \rightarrow [0, 1]$ is ultramodular if and only if $f$ is supermodular and each of its one-dimensional sections is convex, i.e., for each $x \in [0, 1]^n$ and each $i \in \{1, \ldots, n\}$ the function $f_{x,i} : [0, 1] \rightarrow [0, 1]$ given by $f_{x,i}(u) = f(y)$, where $y_i = u$ and $y_j = x_j$ whenever $j \neq i$, is convex.

Corollary

Let $n \geq 2$ and assume that all partial derivatives of order 2 of the function $f : [0, 1]^n \rightarrow [0, 1]$ exist. Then $f$ is ultramodular if and only if all partial derivatives of order 2 are non-negative.
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A function $f : [0, 1]^n \rightarrow [0, 1]$ is ultramodular if and only if $f$ is supermodular and each of its one-dimensional sections is convex, i.e., for each $x \in [0, 1]^n$ and each $i \in \{1, \ldots, n\}$ the function $f_{x,i} : [0, 1] \rightarrow [0, 1]$ given by $f_{x,i}(u) = f(y)$, where $y_i = u$ and $y_j = x_j$ whenever $j \neq i$, is convex.

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Let $n \geq 2$ and assume that all partial derivatives of order 2 of the function $f : [0, 1]^n \rightarrow [0, 1]$ exist. Then $f$ is ultramodular if and only if all partial derivatives of order 2 are non-negative.
For an n-ary aggregation function $A : [0, 1]^n \rightarrow [0, 1]$ the following are equivalent:

(a) $A$ is ultramodular;
(b) each two-dimensional section of $A$ is ultramodular;
(c) each two-dimensional section of $A$ is supermodular and each one-dimensional section of $A$ is convex.
Another equivalent condition to the ultramodularity (8) of an \( n \)-ary aggregation function \( A: [0, 1]^n \rightarrow [0, 1] \) is the validity of

\[
A(x + u) + A(x - u) \geq A(x + v) + A(x - v)
\]

for all \( x, u \in [0, 1]^n, v \in \mathbb{R}^n \) with \( |v| \leq u \) and \( x + u, x - u, x + v, x - v \in [0, 1]^n \) (indeed, it is sufficient to put \( y = u + v \) and \( z = u - v \)). Relaxing the requirement \( u \in [0, 1]^n \) and \( |v| \leq u \) into \( u \in \mathbb{R}^n \) and \( |v| \leq |u| \) we get the definition of \textit{symmetrically monotone functions}. Note that symmetrically monotone aggregation functions \( A: [0, 1]^n \rightarrow [0, 1] \) are exactly ultramodular aggregation functions which are modular, i.e., \( A(x) = \sum_{i=1}^{n} f_i(x_i) \) with \( f_i: [0, 1] \rightarrow [0, 1] \) being convex for each \( i \in \{1, \ldots, n\} \).
For $n = 2$, the ultramodularity (8) of an aggregation function $A: [0, 1]^2 \rightarrow [0, 1]$ is equivalent to $A$ being $P$-increasing, i.e., to

$$A(u_1, v_1) + A(u_4, v_4) \geq \max (A(u_2, v_2) + A(u_3, v_3), A(u_3, v_2) + A(u_2, v_3))$$

for all $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 \in [0, 1]$ satisfying

$u_1 \leq u_2 \land u_3 \leq u_2 \lor u_3 \leq u_4, \ v_1 \leq v_2 \land v_3 \leq v_2 \lor v_3 \leq v_4,$

$u_1 + u_4 \geq u_2 + u_3,$ and $v_1 + v_4 \geq v_2 + v_3.$
Fix two non-decreasing functions $f, g : [0, 1] \to [0, 1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then the smallest supermodular aggregation function $A_* : [0, 1]^2 \to [0, 1]$ satisfying $A_*(x, 1) = f(x)$ and $A_*(1, y) = g(y)$ for all $x, y \in [0, 1]$ is given by $A_*(x, y) = \max(f(x) + g(y) - 1, 0)$. Observe that $A_*$ is ultramodular (and, subsequently, the smallest ultramodular aggregation function with fixed margins $f$ and $g$) if and only if both $f$ and $g$ are convex. In particular, if $f = g = \text{id}_{[0,1]}$ then $A_* = W$, the smallest binary copula. On the other hand, the greatest supermodular aggregation function $A^* : [0, 1]^2 \to [0, 1]$ satisfying $A^*(x, 1) = f(x)$ and $A^*(1, y) = g(y)$ for all $x, y \in [0, 1]$ is given by $A^*(x, y) = \min(f(x), g(y))$. However, $A^*$ is ultramodular only if $f = g = 1 \{1\}$, i.e., if $A^*$ coincides with the smallest binary aggregation function $1 \{(1,1)\}$. 
Given a copula $C : [0, 1]^2 \to [0, 1]$, for each $c \in [0, 1]$ the horizontal section $h_c : [0, 1] \to [0, 1]$ given by $h_c(x) = C(x, c)$ obviously satisfies $h_c(0) = 0$ and $h_c(1) = c$. Then the greatest possible convex horizontal section $h_c$ is given by $h_c(x) = c \cdot x$, corresponding to the product copula $\Pi$ (hence we have $C(x, c) \leq c \cdot x = \Pi(x, c)$). It is easy to verify that $\Pi$ is an ultramodular copula, and hence $\Pi$ is the greatest ultramodular copula. From a statistical point of view, ultramodular copulas describe the dependence structure of stochastically decreasing random vectors $(X, Y)$, and thus each ultramodular copula is *Negative Quadrant Dependent* (NQD).
**Figure:** Modularity (left), supermodularity (center), and ultramodularity of a function $f : [0, 1]^2 \rightarrow [0, 1]$
Theorem

Let $A: [0, 1]^n \rightarrow [0, 1]$ be an aggregation function and $k \geq 2$. Then the following are equivalent:

(i) $A$ is ultramodular.

(ii) If $B_1, \ldots, B_n: [0, 1]^k \rightarrow [0, 1]$ are non-decreasing supermodular functions then the composite $D: [0, 1]^k \rightarrow [0, 1]$ given by $D(x) = A(B_1(x), \ldots, B_n(x))$ is a supermodular function.
Theorem

Let $A: [0, 1]^n \to [0, 1]$ and $B_1, \ldots, B_n: [0, 1]^k \to [0, 1]$ be ultramodular aggregation functions. Then the composite function $D: [0, 1]^k \to [0, 1]$ given by $D(x) = A(B_1(x), \ldots, B_n(x))$ is also an ultramodular aggregation function.
Corollary

Let $A_1, \ldots, A_j: [0, 1]^n \to [0, 1]$ be $n$-ary ultramodular aggregation functions and $f: [0, 1] \to [0, 1]$ a non-decreasing function with $f(0) = 0$ and $f(1) = 1$. Then we have:

(i) Each convex combination of $A_1, \ldots, A_j$ is an $n$-ary ultramodular aggregation function.

(ii) The product of $A_1, \ldots, A_j$ is an $n$-ary ultramodular aggregation function.

(iii) If $A: [0, 1]^n \to [0, 1]$ is an $n$-ary ultramodular aggregation function and $f$ is convex then the composition $f \circ A$ is an $n$-ary ultramodular aggregation function.
Corollary - continue

(iv) If $A : [0, 1]^2 \rightarrow [0, 1]$ is a binary associative ultramodular aggregation function (i.e., $A(x, A(y, z)) = A(A(x, y), z)$ for all $x, y, z \in [0, 1]$) then, for each $k > 2$, the $k$-ary extension of $A$ to $[0, 1]^k$ defined by

$$A(x_1, x_2 \ldots, x_k) = A(x_1, A(x_2, \ldots, A(x_{k-1}, x_k) \ldots))$$

is a $k$-ary ultramodular aggregation function.
**Proposition**

Let $Ch_m : [0, 1]^n \to [0, 1]$ be a Choquet integral-based aggregation function based on a capacity $m$ on $X = \{1, \ldots, n\}$. Then we have:

(i) $Ch_m$ is superadditive, i.e., for all $x, y \in [0, 1]^n$ with $x + y \in [0, 1]^n$ we have

$$Ch_m(x + y) \geq Ch_m(x) + Ch_m(y),$$

if and only if the capacity $m$ is supermodular.

(ii) $Ch_m$ is ultramodular if and only if the capacity $m$ is modular, i.e., $Ch_m$ is a weighted arithmetic mean.
Lemma

An n-ary ultramodular aggregation function $A: [0, 1]^n \rightarrow [0, 1]$ is continuous if and only if $\sup\{A(x) \mid x \in [0, 1]^n\} = 1$.

Proposition

Each function $A \in \mathcal{U}_n$ can be written as a convex combination $A = \lambda A^* + (1 - \lambda)A^{**}$ where $\lambda = \sup\{A(x) \mid x \in [0, 1]^n\}$, $A^* \in \mathcal{U}_n$ is continuous and $A^{**}$ is an n-ary aggregation function with $A^{**}(x) = 0$ for all $x \in [0, 1]^n$. 
Lemma

An n-ary ultramodular aggregation function \( A : [0, 1]^n \rightarrow [0, 1] \) is continuous if and only if \( \sup \{ A(x) \mid x \in [0, 1]^n \} = 1 \).

Proposition

Each function \( A \in \mathcal{U}_n \) can be written as a convex combination \( A = \lambda A^* + (1 - \lambda) A^{**} \) where \( \lambda = \sup \{ A(x) \mid x \in [0, 1]^n \} \), \( A^* \in \mathcal{U}_n \) is continuous and \( A^{**} \) is an n-ary aggregation function with \( A^{**}(x) = 0 \) for all \( x \in [0, 1]^n \).
Theorem

Let $A: [0, 1]^n \rightarrow [0, 1]$ be an aggregation function with $A(x) = 0$ for all $x \in [0, 1]^n$. Then $A$ is ultramodular if and only if the following hold:

(i) all $(n - 1)$-dimensional sections $B_i = A|_{E_i}$ of $A$, $i \in \{1, \ldots, n\}$, are ultramodular, where $E_i = E_{e_i;e_1,e_2,\ldots,e_i-1,e_i+1,\ldots,e_n}$.

(ii) for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and all $x \in E_i \cap E_j$ we have

$$A(x) \geq \sup\{B_i(y) \mid y \in E_i, y < x\} + \sup\{B_j(z) \mid z \in E_j, z < x\}.$$
Proposition

A function $A: [0, 1]^2 \rightarrow [0, 1]$ is a maximal continuous ultramodular aggregation function (i.e., there is no continuous ultramodular aggregation function $B: [0, 1]^2 \rightarrow [0, 1]$ with $B(x, y) \geq A(x, y)$ for all $(x, y) \in [0, 1]^2$ and $B(x_0, y_0) > A(x_0, y_0)$ for some $(x_0, y_0) \in [0, 1]^2$) if and only if $A$ is a weighted arithmetic mean, i.e., $A(x, y) = \lambda \cdot x + (1 - \lambda) \cdot y$ for some $\lambda \in [0, 1]$. 
Corollary

If $A \in \mathcal{U}_2$ then we have

$$A = \lambda \cdot A_1 + (1 - \lambda) \cdot A_2,$$

where $A_1$ is a modular element of $\mathcal{U}_2$, $A_2$ is a supermodular binary aggregation function with annihilator 0, and

$\lambda = 1 - A(1, 0) - A(0, 1) \in [0, 1]$. 

Recall that a binary aggregation function $C : [0, 1]^2 \rightarrow [0, 1]$ is an Archimedean copula if and only if there is a continuous, strictly decreasing convex function $t : [0, 1] \rightarrow [0, \infty]$ with $t(1) = 0$ such that for all $(x, y) \in [0, 1]$

$$C(x, y) = t^{-1}(\min(t(x) + t(y), t(0))). \; \; \; (10)$$

The function $t$ is called an additive generator of $C$.

**Theorem**

Let $C : [0, 1]^2 \rightarrow [0, 1]$ be an Archimedean copula with a two times differentiable additive generator $t : [0, 1] \rightarrow [0, \infty]$. Then $C$ is ultramodular if and only if $\frac{1}{t'}$ is a convex function.
Recall that a binary aggregation function $C : [0, 1]^2 \rightarrow [0, 1]$ is an Archimedean copula if and only if there is a continuous, strictly decreasing convex function $t : [0, 1] \rightarrow [0, \infty]$ with $t(1) = 0$ such that for all $(x, y) \in [0, 1]$

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Let $C : [0, 1]^2 \rightarrow [0, 1]$ be an Archimedean copula with a two times differentiable additive generator $t : [0, 1] \rightarrow [0, \infty]$. Then $C$ is ultramodular if and only if $\frac{1}{t'}$ is a convex function.
For each continuous, convex and strictly decreasing function $f : [0, 1] \to [0, \infty]$ with $f(1) = 0$ (i.e., for each additive generator of an Archimedean copula define the function $C_f : [0, 1]^2 \to [0, 1]$ by

$$C_f(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ x \cdot f^{-1}(\min\left(\frac{f(y)}{x}, f(0)\right)) & \text{otherwise.} \end{cases} \quad (11)$$

and

$$C_f(x, y) = \begin{cases} 0 & \text{if } y = 0, \\ y \cdot f^{-1}(\min\left(\frac{f(x)}{y}, f(0)\right)) & \text{otherwise.} \end{cases} \quad (12)$$

are copulas.
Theorem

Let $f : [0, 1] \to [0, \infty]$ be a two times differentiable horizontal or vertical generator. If $\frac{1}{f'}$ is a convex function then $C_f$ and $C_f^r$ are ultramodular.

However, previous Theorem provides only a sufficient condition for the ultramodularity of copulas: indeed, if $f : [0, 1] \to [0, 1]$ is given by $f(x) = \frac{1}{x} - 1$, then $f$ is two times differentiable and the copulas $C_f$ and $C_f^r$, given by

$$C_f(x, y) = \frac{x^2y}{1 - y + xy}, \quad C_f^r(x, y) = \frac{xy^2}{1 - x + xy},$$

are both ultramodular, but $\frac{1}{f'}$ is not convex (in fact, it is concave).
Theorem

Let \( f : [0, 1] \rightarrow [0, \infty] \) be a two times differentiable horizontal or vertical generator. If \( \frac{1}{f'} \) is a convex function then \( C_f \) and \( C^f \) are ultramodular.

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\[
C_f(x, y) = \frac{x^2 y}{1 - y + xy}, \quad C^f(x, y) = \frac{xy^2}{1 - x + xy},
\]

are both ultramodular, but \( \frac{1}{f'} \) is not convex (in fact, it is concave).
**Theorem**

Let $A: [0, 1]^n \rightarrow [0, 1]$ be a continuous ultramodular aggregation function. Let $C_1, \ldots, C_n: [0, 1]^2 \rightarrow [0, 1]$ be copulas and assume that the continuous non-decreasing functions $f_1, \ldots, f_n, g_1, \ldots, g_n: [0, 1] \rightarrow [0, 1]$ satisfy $f_i(1) = g_i(1) = 1$ for each $i \in \{1, \ldots, n\}$ and $A(f_1(0), \ldots, f_n(0)) = A(g_1(0), \ldots, g_n(0)) = 0$. Define $\xi, \eta: [0, 1] \rightarrow [0, 1]$ by

\[
\xi(x) = \sup\{u \in [0, 1] \mid A(f_1(u), \ldots, f_n(u)) \leq x\}, \\
\eta(x) = \sup\{u \in [0, 1] \mid A(g_1(u), \ldots, g_n(u)) \leq x\}.
\]

Then the function $C: [0, 1]^2 \rightarrow [0, 1]$ given by

\[
C(x, y) = A(C_1(f_1 \circ \xi(x), g_1 \circ \eta(y)), \ldots, C_n(f_n \circ \xi(x), g_n \circ \eta(y)))
\]

is a copula.
If we put \( n = 2 \), \( A = C_1 = \prod \) and define, for \( \alpha, \beta \in [0, 1] \), the functions \( f_1, f_2, g_1, g_2 \) by \( f_1(x) = x^{1-\alpha} \), \( f_2(x) = x^\alpha \), \( g_1(x) = x^{1-\beta} \), and \( g_2(x) = x^\beta \), then for each copula \( C_2 \) the construction in (13) yields the copula \( C_{\alpha,\beta} \) given by

\[
C_{\alpha,\beta} = x^{1-\alpha} \cdot y^{1-\beta} \cdot C_2(x^\alpha, y^\beta).
\]
If we put $n = 2$, $A = W$, $C_1 = C_2 = M$ defined by $M(x, y) = \min(x, y)$ and define the functions $f_1, f_2, g_1, g_2$ by $f_1(x) = g_2(x) = \frac{x+2}{3}$ and $f_2(x) = g_1(x) = \frac{2x+1}{3}$, then the construction in (13) yields the copula $C$ given by

$$C(x, y) = \frac{1}{3} \cdot \max(\min(x + 1, 2y) + \min(2x, y + 1) - 1, 0).$$
Copulas with dimension $> 2$?

stronger forms of ultramodularity are necessary

recall Archimedean copulas and their additive generators!
Thanks for your attention