Aggregation functions and ultramodularity

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Contents

Let I be a subinterval of $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ be a real function.

(i) f is said to be *convex* if, for all $x, y \in I$ and for all $\lambda \in [0, 1]$,

$$\lambda \cdot f(x) + (1 - \lambda) \cdot f(y) \geq f(\lambda \cdot x + (1 - \lambda) \cdot y); \tag{1}$$

(ii) f is said to be *Jensen convex* if, for all $x, y \in I$,

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x + y}{2}\right). \tag{2}$$

Let I be a subinterval of $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ be a real function. Then we have:

(i) f is convex if and only if, for all $x, y \in I$ and for all $\varepsilon > 0$ such that $x < y$ and $y + \varepsilon \in I$,

$$f(y+\varepsilon) - f(y) \geq f(x+\varepsilon) - f(x). \tag{3}$$

(ii) If f is a continuous function then f is convex if and only if it is Jensen convex.

(iii) If f is a monotone function then f is convex if and only if it is Jensen convex.

(iv) If f is a bounded function then f is convex if and only if it is Jensen convex.

Let $(L, \wedge, \vee)$ be a lattice.

(i) A function $f\colon L \to \mathbb{R}$ is called *modular* if, for all $x, y \in L$,

$$f(x \vee y) + f(x \wedge y) = f(x) + f(y). \tag{4}$$

(ii) A function $f\colon L \to \mathbb{R}$ is called *supermodular* if, for all $x, y \in L$,

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y). \tag{5}$$

Proposition

An n-ary aggregation function $A\colon [0,1]^n \to [0,1]$ *is modular if and only if there are non-decreasing functions* $f_1, f_2, \ldots, f_n\colon [0,1] \to [0,1]$ *with* $\sum_{i=1}^n f_i(0) = 0$ *and* $\sum_{i=1}^n f_i(1) = 1$ *such that, for all* $(x_1, \ldots, x_n) \in [0,1]^n$*,*

$$A(x_1, \ldots, x_n) = \sum_{i=1}^n f_i(x_i).$$

For non-decreasing functions $f\colon [0,1]^2 \to [0,1]$, supermodularity can be reformulated as

$$f(x^*, y^*) - f(x^*, y) - f(x, y^*) + f(x, y) \geq 0 \tag{6}$$

for all $x, x^*, y, y^* \in [0,1]$ with $x \leq x^*$ and $y \leq y^*$.

Proposition

An n-ary function $f\colon [0,1]^n \to [0,1]$ is supermodular if and only if each of its two-dimensional sections is supermodular, i.e., for each $\mathbf{x} \in [0,1]^n$ and all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, the function $f_{\mathbf{x},i,j}\colon [0,1]^2 \to [0,1]$ given by $f_{\mathbf{x},i,j}(u, v) = f(\mathbf{y})$, where $y_i = u$, $y_j = v$ and $y_k = x_k$ for $k \in \{1, 2, \ldots, n\} \setminus \{i, j\}$, is supermodular.

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The supermodularity of a function $f\colon [0,1]^n \to [0,1]$ is preserved if the arguments are distorted, i.e., if $g_1, \ldots, g_n\colon [0,1] \to [0,1]$ are non-decreasing functions, then the function $h\colon [0,1]^n \to [0,1]$ given by $h(\mathbf{x}) = f(g_1(x_1), \ldots, g_n(x_n))$ is supermodular (if f is a supermodular aggregation function with $f(g_1(0), \ldots, g_n(0)) = 0$ and $f(g_1(1), \ldots, g_n(1)) = 1$ then h is also a supermodular aggregation function).

Definition

An aggregation function $C\colon [0,1]^2 \to [0,1]$ is called a 2-*copula* (or, briefly, a *copula*) if it is supermodular and has 1 as neutral element, i.e., if $C(x,1) = C(1,x) = x$ for all $x \in [0,1]$.

Proposition

An aggregation function $A\colon [0,1]^2 \to [0,1]$ *is supermodular if and only if there are non-decreasing functions* $g_1, g_2, g_3, g_4\colon [0,1] \to [0,1]$ *with* $g_i(0) = 0$ *and* $g_i(1) = 1$ *for* $i \in \{1,2,3,4\}$*, a copula* $C\colon [0,1]^2 \to [0,1]$*, and numbers* $a, b, c \in [0,1]$ *with* $a + b + c = 1$ *such that, for all* $(x, y) \in [0,1]^2$*,*

$$A(x, y) = a \cdot g_1(x) + b \cdot g_2(y) + c \cdot C(g_3(x), g_4(y)). \tag{7}$$

Definition

An n-ary aggregation function $A\colon [0,1]^n \to [0,1]$ is called *ultramodular* if, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in [0,1]^n$ with $\mathbf{x}+\mathbf{y}+\mathbf{z} \in [0,1]^n$,

$$A(\mathbf{x}+\mathbf{y}+\mathbf{z}) - A(\mathbf{x}+\mathbf{y}) \geq A(\mathbf{x}+\mathbf{z}) - A(\mathbf{x}). \qquad (8)$$

In the case of one-dimensional aggregation functions, ultramodularity is just standard convexity. Therefore, ultramodularity can also be seen as an extension of one-dimensional convexity.

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Proposition

A function $f\colon [0,1]^n \to [0,1]$ is ultramodular if and only if f is supermodular and each of its one-dimensional sections is convex, i.e., for each $\mathbf{x} \in [0,1]^n$ and each $i \in \{1,\ldots,n\}$ the function $f_{\mathbf{x},i}\colon [0,1] \to [0,1]$ given by $f_{\mathbf{x},i}(u) = f(\mathbf{y})$, where $y_i = u$ and $y_j = x_j$ whenever $j \neq i$, is convex.

Corollary

Let $n \geq 2$ and assume that all partial derivatives of order 2 of the function $f\colon [0,1]^n \to [0,1]$ exist. Then f is ultramodular if and only if all partial derivatives of order 2 are non-negative.

Proposition

A function $f\colon [0,1]^n \to [0,1]$ is ultramodular if and only if f is supermodular and each of its one-dimensional sections is convex, i.e., for each $\mathbf{x} \in [0,1]^n$ and each $i \in \{1,\ldots,n\}$ the function $f_{\mathbf{x},i}\colon [0,1] \to [0,1]$ given by $f_{\mathbf{x},i}(u) = f(\mathbf{y})$, where $y_i = u$ and $y_j = x_j$ whenever $j \neq i$, is convex.

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Let $n \geq 2$ and assume that all partial derivatives of order 2 of the function $f\colon [0,1]^n \to [0,1]$ exist. Then f is ultramodular if and only if all partial derivatives of order 2 are non-negative.

For an n-ary aggregation function $A\colon [0,1]^n \to [0,1]$ the following are equivalent:

(a) A is ultramodular;

(b) each two-dimensional section of A is ultramodular;

(c) each two-dimensional section of A is supermodular and each one-dimensional section of A is convex.

Another equivalent condition to the ultramodularity (8) of an n-ary aggregation function $A\colon [0,1]^n \to [0,1]$ is the validity of

$$A(\mathbf{x}+\mathbf{u}) + A(\mathbf{x}-\mathbf{u}) \geq A(\mathbf{x}+\mathbf{v}) + A(\mathbf{x}-\mathbf{v})$$

for all $\mathbf{x},\mathbf{u} \in [0,1]^n$, $\mathbf{v} \in \mathbb{R}^n$ with $|\mathbf{v}| \leq \mathbf{u}$ and $\mathbf{x}+\mathbf{u}, \mathbf{x}-\mathbf{u}, \mathbf{x}+\mathbf{v}, \mathbf{x}-\mathbf{v} \in [0,1]^n$ (indeed, it is sufficient to put $\mathbf{y}=\mathbf{u}+\mathbf{v}$ and $\mathbf{z}=\mathbf{u}-\mathbf{v}$). Relaxing the requirement $\mathbf{u} \in [0,1]^n$ and $|\mathbf{v}| \leq \mathbf{u}$ into $\mathbf{u} \in \mathbb{R}^n$ and $|\mathbf{v}| \leq |\mathbf{u}|$ we get the definition of *symmetrically monotone functions*. Note that symmetrically monotone aggregation functions $A\colon [0,1]^n \to [0,1]$ are exactly ultramodular aggregation functions which are modular, i.e., $A(\mathbf{x}) = \sum_{i=1}^n f_i(x_i)$ with $f_i\colon [0,1] \to [0,1]$ being convex for each $i \in \{1,\dots,n\}$.

For $n = 2$, the ultramodularity (8) of an aggregation function $A\colon [0,1]^2 \to [0,1]$ is equivalent to A being *P-increasing*, i.e., to

$$A(u_1, v_1) + A(u_4, v_4) \geq \max\big(A(u_2, v_2) + A(u_3, v_3), A(u_3, v_2) + A(u_2, v_3)\big)$$

for all $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 \in [0,1]$ satisfying $u_1 \leq u_2 \wedge u_3 \leq u_2 \vee u_3 \leq u_4$, $v_1 \leq v_2 \wedge v_3 \leq v_2 \vee v_3 \leq v_4$, $u_1 + u_4 \geq u_2 + u_3$, and $v_1 + v_4 \geq v_2 + v_3$.

Fix two non-decrasing functions $f, g\colon [0,1] \to [0,1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then the smallest supermodular aggregation function $A_*\colon [0,1]^2 \to [0,1]$ satisfying $A_*(x,1) = f(x)$ and $A_*(1,y) = g(y)$ for all $x, y \in [0,1]$ is given by $A_*(x,y) = \max(f(x) + g(y) - 1, 0)$. Observe that A_* is ultramodular (and, subsequently, the smallest ultramodular aggregation function with fixed margins f and g) if and only if both f and g are convex. In particular, if $f = g = \mathrm{id}_{[0,1]}$ then $A_* = W$, the smallest binary copula. On the other hand, the greatest supermodular aggregation function $A^*\colon [0,1]^2 \to [0,1]$ satisfying $A^*(x,1) = f(x)$ and $A^*(1,y) = g(y)$ for all $x, y \in [0,1]$ is given by $A^*(x,y) = \min(f(x), g(y))$. However, A^* is ultramodular only if $f = g = \mathbf{1}_{\{1\}}$, i.e., if A^* coincides with the smallest binary aggregation function $\mathbf{1}_{\{(1,1)\}}$.

Given a copula $C\colon [0,1]^2 \to [0,1]$, for each $c \in [0,1]$ the horizontal section $h_c\colon [0,1] \to [0,1]$ given by $h_c(x) = C(x,c)$ obviously satisfies $h_c(0) = 0$ and $h_c(1) = c$. Then the greatest possible convex horizontal section h_c is given by $h_c(x) = c \cdot x$, corresponding to the product copula Π (hence we have $C(x,c) \le c \cdot x = \Pi(x,c)$). It is easy to verify that Π is an ultramodular copula, and hence Π is the greatest ultramodular copula. From a statistical point of view, ultamodular copulas describe the dependence structure of stochastically decreasing random vectors (X, Y), and thus each ultramodular copula is *Negative Quadrant Dependent* (NQD).

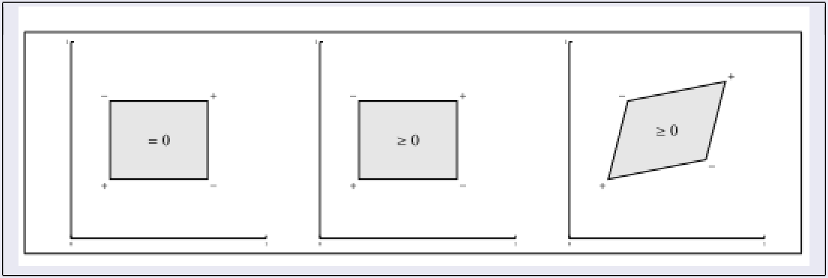

Figure: Modularity (left), supermodularity (center), and ultramodularity of a function $f\colon [0,1]^2 \to [0,1]$

Theorem

Let $A\colon [0,1]^n \to [0,1]$ *be an aggregation function and* $k \geq 2$. *Then the following are equivalent:*

(i) *A is ultramodular.*

(ii) *If* $B_1, \ldots, B_n\colon [0,1]^k \to [0,1]$ *are non-decreasing supermodular functions then the composite* $D\colon [0,1]^k \to [0,1]$ *given by* $D(\mathbf{x}) = A(B_1(\mathbf{x}), \ldots, B_n(\mathbf{x}))$ *is a supermodular function.*

Theorem

Let $A\colon [0,1]^n \to [0,1]$ *and* $B_1,\ldots,B_n\colon [0,1]^k \to [0,1]$ *be ultramodular aggregation functions. Then the composite function* $D\colon [0,1]^k \to [0,1]$ *given by* $D(\mathbf{x}) = A(B_1(\mathbf{x}),\ldots,B_n(\mathbf{x}))$ *is also an ultramodular aggregation function.*

Corollary

Let $A_1, \dots, A_j \colon [0,1]^n \to [0,1]$ be n-ary ultramodular aggregation functions and $f \colon [0,1] \to [0,1]$ a non-decreasing function with $f(0) = 0$ and $f(1) = 1$. Then we have:

(i) Each convex combination of $A_1, \dots, A_j$ is an n-ary ultramodular aggregation function.

(ii) The product of $A_1, \dots, A_j$ is an n-ary ultramodular aggregation function.

(iii) If $A \colon [0,1]^n \to [0,1]$ is an n-ary ultramodular aggregation function and f is convex then the composition $f \circ A$ is an n-ary ultramodular aggregation function.

Corollary - continue

(iv) If $A\colon [0,1]^2 \to [0,1]$ is a binary associative ultramodular aggregation function (i.e., $A(x, A(y,z)) = A(A(x,y),z)$ for all $x, y, z \in [0,1]$) then, for each $k > 2$, the k-ary extension of A to $[0,1]^k$ defined by

$$A(x_1, x_2 \ldots, x_k) = A(x_1, A(x_2, \ldots, A(x_{k-1}, x_k) \ldots))$$

is a k-ary ultramodular aggregation function.

Proposition

Let $\mathrm{Ch}_m\colon [0,1]^n \to [0,1]$ *be a Choquet integral-based aggregation function based on a capacity* m *on* $X = \{1, \ldots, n\}$*. Then we have:*

(i) Ch_m *is superadditive, i.e., for all* $\mathbf{x}, \mathbf{y} \in [0,1]^n$ *with* $\mathbf{x} + \mathbf{y} \in [0,1]^n$ *we have*

$$\mathrm{Ch}_m(\mathbf{x} + \mathbf{y}) \geq \mathrm{Ch}_m(\mathbf{x}) + \mathrm{Ch}_m(\mathbf{y}),$$

if and only if the capacity m *is supermodular.*

(ii) Ch_m *is ultramodular if and only if the capacity* m *is modular, i.e.,* Ch_m *is a weighted arithmetic mean.*

Lemma

An n-ary ultramodular aggregation function $A\colon [0,1]^n \to [0,1]$ *is continuous if and only if* $\sup\{A(\mathbf{x}) \mid \mathbf{x} \in [0,1[^n\} = 1$.

Proposition

Each function $A \in \mathcal{U}_n$ *can be written as a convex combination* $A = \lambda A^* + (1-\lambda)A^{**}$ *where* $\lambda = \sup\{A(\mathbf{x}) \mid \mathbf{x} \in [0,1[^n\}$, $A^* \in \mathcal{U}_n$ *is continuous and* A^{**} *is an n-ary aggregation function with* $A^{**}(\mathbf{x}) = 0$ *for all* $\mathbf{x} \in [0,1[^n$.

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Theorem

Let $A\colon [0,1]^n \to [0,1]$ *be an aggregation function with* $A(\mathbf{x}) = 0$ *for all* $\mathbf{x} \in [0,1[^n$. *Then* A *is ultramodular if and only if the following hold:*

(i) *all* $(n-1)$*-dimensional sections* $B_i = A|_{E_i}$ *of* A, $i \in \{1,\ldots,n\}$, *are ultramodular, where* $E_i = E_{\mathbf{e}_i;\mathbf{e}_1,\ldots,\mathbf{e}_{i-1},\mathbf{e}_{i+1},\ldots,\mathbf{e}_n}$.

(ii) *for all* $i,j \in \{1,\ldots,n\}$ *with* $i \neq j$ *and all* $\mathbf{x} \in E_i \cap E_j$ *we have*

$$A(\mathbf{x}) \geq \sup\{B_i(\mathbf{y}) \mid \mathbf{y} \in E_i, \mathbf{y} < \mathbf{x}\} + \sup\{B_j(\mathbf{z}) \mid \mathbf{z} \in E_j, \mathbf{z} < \mathbf{x}\}.$$

Proposition

A function $A\colon [0,1]^2 \to [0,1]$ *is a maximal continuous ultramodular aggregation function (i.e., there is no continuous ultramodular aggregation function* $B\colon [0,1]^2 \to [0,1]$ *with* $B(x,y) \geq A(x,y)$ *for all* $(x,y) \in [0,1]^2$ *and* $B(x_0,y_0) > A(x_0,y_0)$ *for some* $(x_0,y_0) \in [0,1]^2$*) if and only if A is a weighted arithmetic mean, i.e.,* $A(x,y) = \lambda \cdot x + (1-\lambda) \cdot y$ *for some* $\lambda \in [0,1]$.

Corollary

If $A \in \mathcal{U}_2$ then we have

$$A = \lambda \cdot A_1 + (1 - \lambda) \cdot A_2, \tag{9}$$

where A_1 is a modular element of $\mathcal{U}_2$, A_2 is a supermodular binary aggregation function with annihilator 0, and $\lambda = 1 - A(1,0) - A(0,1) \in [0,1]$.

Recall that a binary aggregation function $C\colon [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a continuous, strictly decreasing convex function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$ such that for all $(x,y) \in [0,1]$

$$C(x,y) = t^{-1}(\min(t(x)+t(y), t(0))). \tag{10}$$

The function t is called an additive generator of C.

Theorem

Let $C\colon [0,1]^2 \to [0,1]$ be an Archimedean copula with a two times differentiable additive generator $t\colon [0,1] \to [0,\infty]$. Then C is ultramodular if and only if $\frac{1}{t'}$ is a convex function.

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For each continuous, convex and strictly decreasing function $f\colon [0,1] \to [0,\infty]$ with $f(1) = 0$ (i.e., for each additive generator of an Archimedean copula define the function $C_f\colon [0,1]^2 \to [0,1]$ by

$$C_f(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ x \cdot f^{-1}\left(\min\left(\frac{f(y)}{x}, f(0)\right)\right) & \text{otherwise.} \end{cases} \tag{11}$$

and

$$C^f(x,y) = \begin{cases} 0 & \text{if } y = 0, \\ y \cdot f^{-1}\left(\min\left(\frac{f(x)}{y}, f(0)\right)\right) & \text{otherwise.} \end{cases} \tag{12}$$

are copulas.

Theorem

Let $f\colon [0,1] \to [0,\infty]$ be a two times differentiable horizontal or vertical generator. If $\frac{1}{f'}$ is a convex function then C_f and C^f are ultramodular.

However, previous Theorem provides only a sufficient condition for the ultramodularity of copulas: indeed, if $f\colon [0,1] \to [0,1]$ is given by $f(x) = \frac{1}{x} - 1$, then f is two times differentiable and the copulas C_f and C^f, given by

$$C_f(x,y) = \frac{x^2 y}{1 - y + xy}, \qquad C^f(x,y) = \frac{xy^2}{1 - x + xy},$$

are both ultramodular, but $\frac{1}{f'}$ is not convex (in fact, it is concave).

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are both ultramodular, but $\frac{1}{f'}$ is not convex (in fact, it is concave).

Theorem

Let $A\colon [0,1]^n \to [0,1]$ *be a continuous ultramodular aggregation function. Let* $C_1,\ldots,C_n\colon [0,1]^2 \to [0,1]$ *be copulas and assume that the continuous non-decreasing functions* $f_1,\ldots,f_n,g_1,\ldots,g_n\colon [0,1] \to [0,1]$ *satisfy* $f_i(1) = g_i(1) = 1$ *for each* $i \in \{1,\ldots,n\}$ *and* $A(f_1(0),\ldots,f_n(0)) = A(g_1(0),\ldots,g_n(0)) = 0$*. Define* $\xi,\eta\colon [0,1] \to [0,1]$ *by*

$$\xi(x) = \sup\{u \in [0,1] \mid A(f_1(u),\ldots,f_n(u)) \le x\},$$
$$\eta(x) = \sup\{u \in [0,1] \mid A(g_1(u),\ldots,g_n(u)) \le x\}.$$

Then the function $C\colon [0,1]^2 \to [0,1]$ *given by*

$$C(x,y) = A\big(C_1(f_1\circ\xi(x), g_1\circ\eta(y)),\ldots,C_n(f_n\circ\xi(x), g_n\circ\eta(y))\big) \quad (13)$$

is a copula.

If we put $n = 2$, $A = C_1 = \Pi$ and define, for $\alpha, \beta \in [0, 1]$, the functions f_1, f_2, g_1, g_2 by $f_1(x) = x^{1-\alpha}$, $f_2(x) = x^{\alpha}$, $g_1(x) = x^{1-\beta}$, and $g_2(x) = x^{\beta}$, then for each copula C_2 the construction in (13) yields the copula $C_{\alpha,\beta}$ given by

$$C_{\alpha,\beta} = x^{1-\alpha} \cdot y^{1-\beta} \cdot C_2(x^{\alpha}, y^{\beta}).$$

If we put $n = 2$, $A = W$, $C_1 = C_2 = M$ defined by $M(x, y) = \min(x, y)$ and define the functions f_1, f_2, g_1, g_2 by $f_1(x) = g_2(x) = \frac{x+2}{3}$ and $f_2(x) = g_1(x) = \frac{2x+1}{3}$, then the construction in (13) yields the copula C given by

$$C(x, y) = \frac{1}{3} \cdot \max(\min(x + 1, 2y) + \min(2x, y + 1) - 1, 0).$$

Copulas with dimension > 2?

stronger forms of ultramodularity are necessary

recall Archimedean copulas and their additive generators!

Thanks for your attention