On completion of Archimedean atomic lattice effect algebras

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Effect algebras

Definition

$(E; \oplus, \mathbf{0}, \mathbf{1})$ is an **effect algebra** if $\oplus$ is a partially defined operation s.t. for all $p, q, r \in E$:

(E1) $p \oplus q = q \oplus p$.

(E2) $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(E3) For every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q = \mathbf{1}$.

(E4) If $p \oplus 1$ is defined then $p = \mathbf{0}$.

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Definition

Let $a, b \in E$. $a \leq b$ if there is a $c \in E$ such that $a \oplus c = b$.
If $(E, \leq)$ is a lattice then E is a lattice effect algebra.

Assume that E is an effect algebra.

An element $a \in E$ is called *atom* if for all $\mathbf{0} \neq b \in E$, $b \leq a$ implies $b = a$.

L is called *atomic* if for each element $\mathbf{0} \neq b \in E$ there exists an atom a such that $a \leq b$.

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To each $a \in E$ we define its isotropic index, $ord(a)$ as the maximal positive integer n such that

$$na = \underbrace{a \oplus a \oplus \cdots \oplus a}_{n-\text{times}}.$$

E is said to be Archimedean if for each $a \in E$, $ord(a)$ is finite.

Further notions

An element $a \in E$ is sharp if

$$a \wedge a' = \mathbf{0}$$

Elements $a, b \in E$ are compatible if

$$a \vee b = a \oplus (b \ominus (b \wedge a))$$

A subset $B \subset E$ is called a block of E if it is a maximal set of pairwise compatible elements. A block is an MV-effect algebra.

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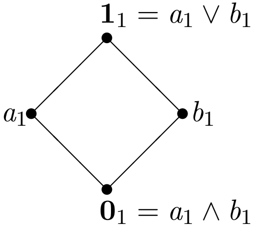
Let E be an atomic Archimedean lattice effect algebra. Then $e \leq f'$ if

1. there exists an atomic block $B \subset E$ such that $e, f \in B$,
2. for an arbitrary atom $a \in B$ if $c_e a \leq e$ and $c_f a \leq f$, then $c_e + c_f \leq ord(a)$.

Z. Riečanová: Lattice effect algebras densely embeddable into complete ones. *Kybernetika* **47** (2011), 100–109.

There are atomic bounded lattices which can be equipped with several effect-algebraic operations



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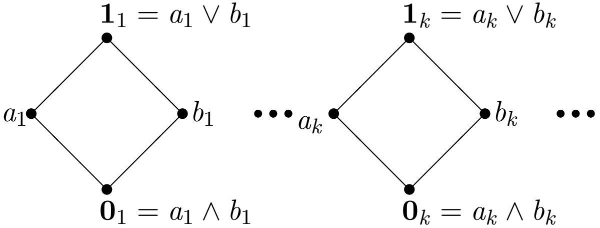

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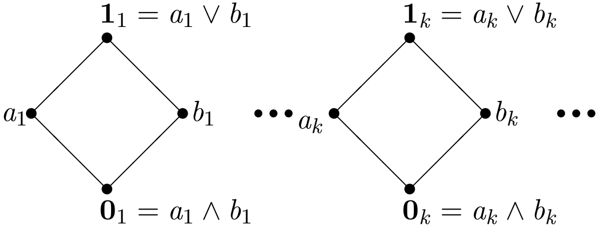

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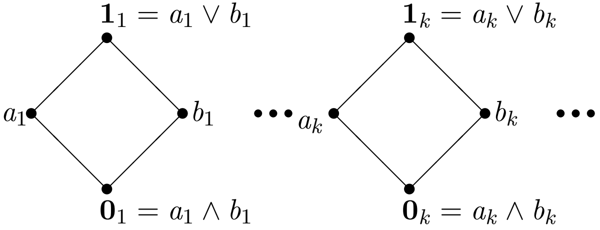

$\mathbf{1}_k = a_k \oplus b_k$ (a_k, b_k are compatible),

$\mathbf{1}_k = 2a_k = 2b_k$ (a_k, b_k are non-compatible).

Open problem by Riečanová

Assume that $(E, \oplus, \mathbf{0}, \mathbf{1})$ is an atomic Archimedean lattice effect algebra such that an effect-algebraic operation $\tilde{\oplus}$ exists onto $\mathcal{MC}(E)$ (the MacNeille completion of E). Does there exist an effect-algebraic operation $\hat{\oplus}$ onto $\mathcal{MC}(E)$ that extends the operation $\oplus$?

MacNeille completion

The MacNeille completion $\hat{E}$ of an effect algebra E is complete lattice into which E can be embeded preserving operations $\vee, \wedge$, such that $\forall c \in \hat{E}$ there exist sequences $\{e_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty}$ of elements from E for which

$$\bigvee_i e_i = \bigwedge_i f_i = c$$

Moreover we assume that the partial operation $\oplus$ can be extended to $\hat{E}$.

Strongly D-continuous effect algebras: Riečanová, MacNeille completions of D-posets and effect algebras, IJTP 39 (2000), 859–869.

Notation:
$\emptyset \neq U \subset E$, $\underline{\mathsf{U}} = \{q \in E; (\forall u \in U)(q \leq u)\}$,
$\emptyset \neq V \subset \underline{\mathsf{U}}$, $\bar{V} = \{\bar{q} \in E; (\forall v \in V)(\bar{q} \geq v)\}$,
$D(U, V) = \{u \ominus v; u \in U, \; v \in V\}$

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Definition

An effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is called strongly D-continuous if for all U, V

$$(\mathrm{SDC}) \qquad \wedge D(U, V) = \mathbf{0} \quad \Leftrightarrow \quad (\forall \bar{v} \in \bar{V})(\forall \underline{\mathsf{u}} \in \underline{\mathsf{U}})(\bar{v} \geq \underline{\mathsf{u}}).$$

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Theorem

An arbitrary effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ has a MacNeille completion iff E has the property (SDC).

(SDC) $\wedge D(U, V) = \mathbf{0} \quad \Leftrightarrow \quad \bar{V} \geq \underline{\mathsf{U}}$

Theorem

Let E an atomic Archimedean LEA. Then E has the property (SDC).

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Let E an atomic Archimedean LEA. Then E has the property (SDC).

Idea of the proof. $\mathcal{A}$ – the set of all atoms.

"$\Rightarrow$" Let $(\exists a \in \mathcal{A}, n \in \mathbf{N})(n \cdot a \in \underline{\mathsf{U}})$

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Corollary

Let E an atomic Archimedean LEA. Then E has $\mathcal{MC}(E)$.