Ortholattices from graphs

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The disclaimer

This in an expository talk about results of other people.

The Papers

L. Lovász:*Kneser's conjecture, chromatic number and homotopy*, J. Combin. Theory Series A **25** (1978), 319–324

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- L. Lovász: *Kneser's conjecture, chromatic number and homotopy*, J. Combin. Theory Series A **25** (1978), 319–324
- J.W. Walker: *From graphs to ortholattices and equivariant maps*, J. Combin. Theory Series B **35** (1983), 171–192

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- $\mathcal{L}(G) = (N(2^V), \cap, \vee, N)$ is an ortholattice, $A \vee B = N^2(A \cup B)$.

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- For an OML, we can take atoms.

From ortholattices to topological spaces

- Take an ortholattice L.
- Remove the top and bottom, denote the resulting poset by $\hat{L}$.
- Replace every n-chain in $\hat{L}$ by an n-simplex and glue the simplices together so that subchains correspond to faces.
- We obtain a topological space $\Delta(\hat{L})$, called *the order complex of* $\hat{L}$.
- The orthocomplementation on $\hat{L}$ can be transferred to a free action of Z_2 on $\Delta(\hat{L})$.