Ortholattices from graphs

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This is an expository talk about results of other people.
The Papers


From graphs to ortholattices

- Take a loopless, undirected graph $G = (V, E)$. 

- $N: V \rightarrow 2^V$ is the neighbourhood map: $N(a) = \{ b : \text{there is an edge } (a, b) \in E \}$.

- Make a natural extension $N: 2^V \rightarrow 2^V$ $N(A) = \bigcap_{a \in A} N(a)$.

- $N$ is antitone.

- No loops: $N(A) \cap A = \emptyset$, $N(\emptyset) = V$, $N(V) = \emptyset$.

- Define: $N(2^V) = \{ N(A) : A \subseteq V \}$ are the closed sets.

- Fact: $N^3 = N$, so $N$ is an antitone involution on the poset of closed sets.

- $L(G) = (N(2^V), \cap, \lor, N)$ is an ortholattice, $A \lor B = N(2^V \cup A \cup B)$. 

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- $L(G) = (N(2^V), \cap, \lor, N)$ is an ortholattice,
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From ortholattices to graphs

- Take an ortholattice $L$.
- Pick a subset $T \subseteq L$ such that every element of $L$ is a join of some elements from $T$.
- Construct a graph $G(L, T)$:
  - the vertices are $T$,
  - the edges are $\{(a, b) \in T \times T : a \leq b'\}$.
- Then we have $L(G(L, T)) \simeq L$.
- Moreover, if $G$ is a graph such that $L(G) \simeq L$, then the neighbourhood retract of $G$ is isomorphic to some $G(L, T)$.
- For an OML, we can take atoms.
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From ortholattices to topological spaces

- Take an ortholattice $L$.
- Remove the top and bottom, denote the resulting poset by $\hat{L}$.
- Replace every $n$-chain in $\hat{L}$ by an $n$-simplex and glue the simplices together so that subchains correspond to faces.
- We obtain a topological space $\Delta(\hat{L})$, called the order complex of $\hat{L}$.
- The orthocomplementation on $\hat{L}$ can be transferred to a free action of $\mathbb{Z}_2$ on $\Delta(\hat{L})$. 