Convexity and weak convexity

Vladimír Janiš*, Tania Iglesias** and Ignacio Montes**

*Matej Bel University, Slovakia
**University of Oviedo, Spain

FSTA 2012, Liptovský Ján, 31.1.2012

Convex fuzzy set

A fuzzy set μ is said to be convex, if for all $x, y \in \operatorname{supp} \mu$ and $\lambda \in [0, 1]$ there is

$$\mu(\lambda x + (1 - \lambda)y) \geq \lambda\mu(x) + (1 - \lambda)\mu(y).$$

Convex fuzzy set

A fuzzy set μ is said to be convex, if for all $x, y \in \operatorname{supp} \mu$ and $\lambda \in [0, 1]$ there is

$$\mu(\lambda x + (1 - \lambda)y) \geq \lambda\mu(x) + (1 - \lambda)\mu(y).$$

Problems:

Convex fuzzy set

A fuzzy set μ is said to be convex, if for all $x, y \in \operatorname{supp} \mu$ and $\lambda \in [0, 1]$ there is

$$\mu(\lambda x + (1-\lambda)y) \geq \lambda\mu(x) + (1-\lambda)\mu(y).$$

Problems:

- not suitable for the case of a lattice valued fuzzy set

Convex fuzzy set

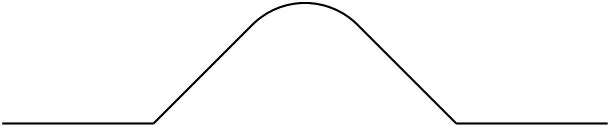
A fuzzy set μ is said to be convex, if for all $x, y \in \operatorname{supp} \mu$ and $\lambda \in [0, 1]$ there is

$$\mu(\lambda x + (1-\lambda)y) \geq \lambda\mu(x) + (1-\lambda)\mu(y).$$

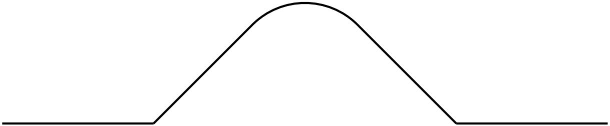
Problems:

- not suitable for the case of a lattice valued fuzzy set
- a convex fuzzy set has all its cuts convex, but NOT vice versa

Two examples

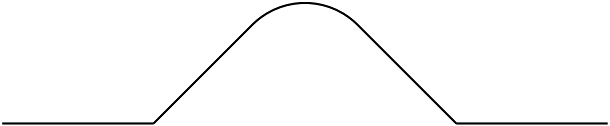


Two examples

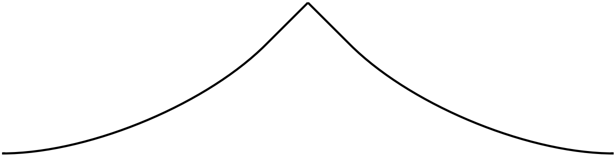


- convex, all cuts are convex

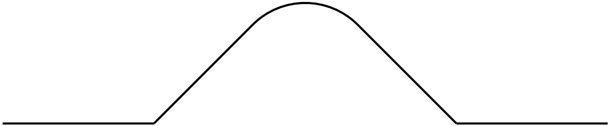
Two examples



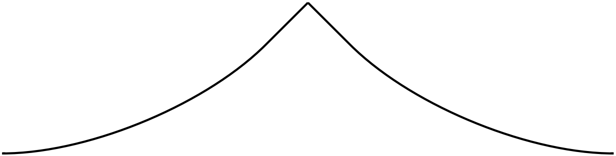
- convex, all cuts are convex



Two examples



- convex, all cuts are convex



- not convex, but the cuts are still convex

Quasiconvex fuzzy sets

Definition

A fuzzy set $\mu : R^n \to R$ is quasiconvex if for all $x, y \in R^n, \lambda \in [0, 1]$ there is $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$.

Definition

A fuzzy set $\mu : R^n \to R$ is quasiconvex if for all $x, y \in R^n, \lambda \in [0,1]$ there is $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$.

The class of all quasiconvex fuzzy real valued sets is exactly the class of those fuzzy sets, for which all their cuts are convex. The following proposition claims that the same holds also for lattice-valued fuzzy sets.

Definition

A fuzzy set $\mu : R^n \to R$ is quasiconvex if for all $x, y \in R^n, \lambda \in [0, 1]$ there is $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$.

The class of all quasiconvex fuzzy real valued sets is exactly the class of those fuzzy sets, for which all their cuts are convex. The following proposition claims that the same holds also for lattice-valued fuzzy sets.

Proposition on cuts

Let μ be an lattice-valued fuzzy sets on R^n. Then μ_α is convex for all $\alpha \in L$ if and only if μ is quasiconvex.

Convexity of crisp sets - preserved under intersections

Preserving quasiconvexity

Convexity of crisp sets - preserved under intersections

Janiš, V., Král', P., Renčová, M.: Aggregation operators preserving quasiconvexity, Information Sciences, submitted

- the same question of preserving quasiconvexity for the case of quasiconvex lattice-valued fuzzy sets

Preserving quasiconvexity

Convexity of crisp sets - preserved under intersections

Janiš, V., Kráľ, P., Renčová, M.: Aggregation operators preserving quasiconvexity, Information Sciences, submitted

- the same question of preserving quasiconvexity for the case of quasiconvex lattice-valued fuzzy sets
- instead of an intersection we consider any type of a binary aggregation between fuzzy sets

Preserving quasiconvexity

Convexity of crisp sets - preserved under intersections

Janiš, V., Kráľ, P., Renčová, M.: Aggregation operators preserving quasiconvexity, Information Sciences, submitted

- the same question of preserving quasiconvexity for the case of quasiconvex lattice-valued fuzzy sets
- instead of an intersection we consider any type of a binary aggregation between fuzzy sets

Aggregation operators

Let L be a bounded lattice. A mapping $A : L^2 \to L$ is a binary aggregation operator on L, if $A(0,0) = 0, A(1,1) = 1$ and A is monotone in both variables.

Preserving quasiconvexity

Convexity of crisp sets - preserved under intersections

Janiš, V., Kráľ, P., Renčová, M.: Aggregation operators preserving quasiconvexity, Information Sciences, submitted

- the same question of preserving quasiconvexity for the case of quasiconvex lattice-valued fuzzy sets
- instead of an intersection we consider any type of a binary aggregation between fuzzy sets

Aggregation operators

Let L be a bounded lattice. A mapping $A : L^2 \to L$ is a binary aggregation operator on L, if $A(0,0) = 0, A(1,1) = 1$ and A is monotone in both variables.

A is symmetric if $A(x,y) = A(y,x)$ for all $x, y \in L$.

Commuting

Let A, B be arbitrary binary aggregation operators. Them A commutes with B, if for each x_1, x_2, y_1, y_2 there is

$$A(B(x_1, y_1), B(x_2, y_2)) = B(A(x_1, x_2), A(y_1, y_2)).$$

Preserving quasiconvexity - results

Commuting

Let A, B be arbitrary binary aggregation operators. Them A commutes with B, if for each x_1, x_2, y_1, y_2 there is

$$A(B(x_1, y_1), B(x_2, y_2)) = B(A(x_1, x_2), A(y_1, y_2)).$$

Aggregations preserving quasiconvexity

Let $A : L^2 \to L$ be an aggregation operator on a bounded lattice L, let $\mu, \nu : R^n \to L$ be quasiconvex sets. Then the following are equivalent:

- $A(\mu(x), \nu(x))$ is quasiconvex,
- A commutes with the $\wedge$ operator, i.e.
$$A(\alpha \wedge \beta, \gamma \wedge \delta) = A(\alpha, \gamma) \wedge A(\beta, \delta).$$

Preserving quasiconvexity - results

$$A(\alpha, \beta) = A(\beta, \alpha)$$

Preserving quasiconvexity - results

$A(\alpha, \beta) = A(\beta, \alpha)$

Case of symmetric aggregation

Let $A : L^2 \to L$ be a symmetric aggregation operator on a bounded lattice L, let $\mu, \nu : R^n \to L$ be quasiconvex sets. Then the following are equivalent:

- The lattice-valued fuzzy set $A(\mu, \nu)$ is quasiconvex,
- $A(\alpha, \beta) = A(\alpha, \alpha) \wedge A(\beta, \beta) = A(\alpha \wedge \beta, \alpha \wedge \beta)$ for each $\alpha, \beta \in L$.

$A(\alpha,\beta) = A(\beta,\alpha)$

Case of symmetric aggregation

Let $A : L^2 \to L$ be a symmetric aggregation operator on a bounded lattice L, let $\mu, \nu : R^n \to L$ be quasiconvex sets. Then the following are equivalent:

- The lattice-valued fuzzy set $A(\mu,\nu)$ is quasiconvex,
- $A(\alpha,\beta) = A(\alpha,\alpha) \wedge A(\beta,\beta) = A(\alpha \wedge \beta, \alpha \wedge \beta)$ for each $\alpha,\beta \in L$.

Corollary for t-norms

The only t-norm preserving quasiconvexity is the minimum.

Corrollary for real valued mappings:

Preserving quasiconvexity for [0,1]-valued fuzzy sets

Corrollary for real valued mappings:

Real aggregation operators

Let $A : [0,1]^2 \to [0,1]$ be an aggregation operator, let μ, ν be quasiconvex sets on the real line. Then the following are equivalent:

- $A(\mu(x), \nu(x))$ is quasiconvex,
- $A(\alpha, \beta) = A(\alpha \wedge \beta, \alpha \wedge \beta)$ for each $\alpha, \beta \in [0,1]$.

Corrollary for real valued mappings:

Real aggregation operators

Let $A : [0,1]^2 \to [0,1]$ be an aggregation operator, let μ, ν be quasiconvex sets on the real line. Then the following are equivalent:

- $A(\mu(x), \nu(x))$ is quasiconvex,
- $A(\alpha, \beta) = A(\alpha \wedge \beta, \alpha \wedge \beta)$ for each $\alpha, \beta \in [0,1]$.

Symmetric real aggregation operators

Let $A : L^2 \to L$ be a symmetric aggregation operator on a bounded lattice $(L, \leq, 0_L, 1_L)$, let $\mu, \nu : R^n \to L$ be quasiconvex sets. Then the following are equivalent:

- The lattice-valued fuzzy set $A(\mu, \nu)$ is quasiconvex,
- $A(\alpha, \beta) = A(\alpha, \alpha) \wedge A(\beta, \beta) = A(\alpha \wedge \beta, \alpha \wedge \beta)$ for each $\alpha, \beta \in L$.

Preserving quasiconvexity for [0,1]-valued fuzzy sets

General case

The aggregation operator $A : [0,1]^2 \to [0,1]$ preserves quasiconvexity if and only if $A(x,y) = \min\{f_1(x), f_2(y)\}$, where $f_1, f_2 : [0,1] \to [0,1]$ are both nondecreasing, $\min\{f_1(0), f_2(0)\} = 0, f_1(1) = f_2(1) = 1$.

Preserving quasiconvexity for [0,1]-valued fuzzy sets

General case

The aggregation operator $A : [0,1]^2 \to [0,1]$ preserves quasiconvexity if and only if $A(x,y) = \min\{f_1(x), f_2(y)\}$, where $f_1, f_2 : [0,1] \to [0,1]$ are both nondecreasing, $\min\{f_1(0), f_2(0)\} = 0$, $f_1(1) = f_2(1) = 1$.

Symmetric case

Let $A : [0,1]^2 \to [0,1]$ be a symmetric aggregation operator, let μ, ν be quasiconvex fuzzy sets on the real line. Then the following are equivalent:

- The fuzzy set $A(\mu, \nu)$ is quasiconvex,
- $A(\alpha, \alpha) = A(\alpha, 1)$ for each $\alpha \in [0,1]$.

- the cuts of such aggregation operators on $[0,1]^2$ are squares $[\alpha, 1]^2$.

T-convexity

- back to the definition of quasiconvexity

Definition

A fuzzy set $\mu : R^n \to R$ is quasiconvex if for all $x, y \in R^n, \lambda \in [0,1]$ there is $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$.

- back to the definition of quasiconvexity

Definition

A fuzzy set $\mu : R^n \to R$ is quasiconvex if for all $x, y \in R^n, \lambda \in [0, 1]$ there is $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$.

- a straightforward generalization

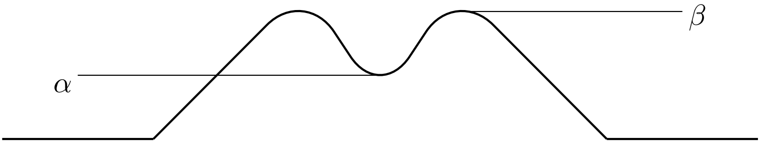
T-convex fuzzy sets

Let μ be a L-valued fuzzy set and let T be a t-norm. Then μ is T-convex if

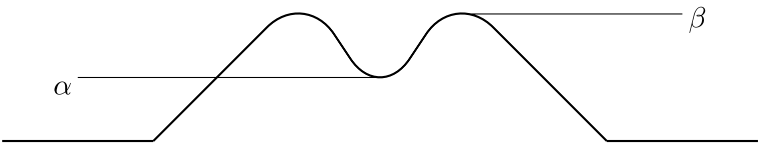
$$\mu(\lambda x + (1 - \lambda)y) \geq T(\mu(x), \mu(y))$$

for every $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Example



Example



T_P -convex under assumption $\alpha \geq \beta^2$

Properties of T-convex fuzzy sets

- if $T_M(x, y) = x \wedge y$, then T_M-convexity of the L-valued fuzzy set μ is equivalent to its quasiconvexity

Properties of T-convex fuzzy sets

- if $T_M(x, y) = x \wedge y$, then T_M-convexity of the L-valued fuzzy set μ is equivalent to its quasiconvexity
- if an L-valued fuzzy set is T-convex, then it is also T^*-convex for every t-norm T^* that satisfies $T^* \leq T$

Properties of T-convex fuzzy sets

- if $T_M(x, y) = x \wedge y$, then T_M-convexity of the L-valued fuzzy set μ is equivalent to its quasiconvexity
- if an L-valued fuzzy set is T-convex, then it is also T^*-convex for every t-norm T^* that satisfies $T^* \leq T$
- if an L-valued fuzzy set is quasiconvex, it is T-convex with respect to every t-norm T

Properties of T-convex fuzzy sets

- if $T_M(x,y) = x \wedge y$, then T_M-convexity of the L-valued fuzzy set μ is equivalent to its quasiconvexity
- if an L-valued fuzzy set is T-convex, then it is also T^*-convex for every t-norm T^* that satisfies $T^* \leq T$
- if an L-valued fuzzy set is quasiconvex, it is T-convex with respect to every t-norm T

Example:

$$\mu(x) = \begin{cases} 0.5 & \text{if } 0 \leq x \leq 2 \text{ or } 4 < x \leq 6 \\ 0.4 & \text{if } 2 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

is T_P-convex, and therefore T_L-convex and T_D-convex, but it is not T_M-convex

Properties of T-convex fuzzy sets

Each T-convex normal fuzzy set is quasiconvex.

Properties of T-convex fuzzy sets

Each T-convex normal fuzzy set is quasiconvex.

Convexity of the support

Let T be a t-norm. The following are equivalent:

- For any T-convex fuzzy set μ the set $\operatorname{supp}\mu$ is convex.
- T has no zero divisors.

Preserving T-convexity = domination of T

Let $A : L^2 \to L$ be an aggregation operator on a bounded lattice L and let $\mu, \nu : R^n \to L$ be T-convex fuzzy sets. Then the following statements are equivalent:

- The lattice valued fuzzy set $A(\mu, \nu)$ is T-convex.
- $A(T(\alpha, \gamma), T(\beta, \delta)) \geq T(A(\alpha, \beta), A(\gamma, \delta))$, for every $\alpha, \beta, \gamma, \delta \in L$, i.e. A dominates T.

Preserving T-convexity = domination of T

Let $A : L^2 \to L$ be an aggregation operator on a bounded lattice L and let $\mu, \nu : R^n \to L$ be T-convex fuzzy sets. Then the following statements are equivalent:

- The lattice valued fuzzy set $A(\mu, \nu)$ is T-convex.
- $A(T(\alpha, \gamma), T(\beta, \delta)) \geq T(A(\alpha, \beta), A(\gamma, \delta))$, for every $\alpha, \beta, \gamma, \delta \in L$, i.e. A dominates T.

Corollary

The T-intersection of two T-convex fuzzy sets is always a T-convex fuzzy set, for any t-norm T.

Syau, Y. R: Some properties of weakly convex fuzzy mappings, Fuzzy Sets and Systems 123, 2001, 203–207.

Weak quasiconvexity

Syau, Y. R: Some properties of weakly convex fuzzy mappings, Fuzzy Sets and Systems 123, 2001, 203–207.

Definition

A fuzzy subset μ of a linear space is weakly quasiconvex if for all $x, y \in \operatorname{supp} \mu$ there exists $\lambda \in (0, 1)$ such that

$$\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}.$$

Weak quasiconvexity

Syau, Y. R: Some properties of weakly convex fuzzy mappings, Fuzzy Sets and Systems 123, 2001, 203–207.

Definition

A fuzzy subset μ of a linear space is weakly quasiconvex if for all $x, y \in \operatorname{supp} \mu$ there exists $\lambda \in (0, 1)$ such that

$$\mu(\lambda x + (1 - \lambda) y) \geq \min\{\mu(x), \mu(y)\}.$$

Example: Dirichlet function

Preserving weak quasiconvexity

Proposition

The only symmetric aggregation operator A such that for any weakly quasiconvex fuzzy sets μ, ν the fuzzy set $A(\mu, \nu)$ is weakly quasiconvex is the mapping $A(0,0) = 0$ and $A(\alpha, \beta) = 1$ otherwise.