Equations and Inequalities Defined by Residuated Functions

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Eleventh International Conference on Fuzzy Set Theory and Applications
January 30-February 3, 2012, Liptovský Ján, Slovak Republic

Functions

Composition of functions

U – non-empty set; I_U – the ***identity function*** on U

U, V, W – non-empty sets; $f : U \to V$, $g : V \to W$ – given functions

$f \circ g : U \to W$ – the ***composition*** of f and g defined by $f \circ g(x) = g(f(x))$, for every $x \in U$

Blyth (2005) and Roman (2008) – the composition is defined in a different way

Direct and inverse images

$f : U \to V$ – function;

≫ $f(S) = \{f(x) \mid x \in S\}$ – the ***direct image*** of $S \subseteq U$ under f;

≫ $f^{-1}(T) = \{x \in U \mid f(x) \in T\}$ – the ***inverse image*** of $T \subseteq V$ under f

Ordered sets

Partially ordered set (ordered set)

$(P, \leqslant)$ – P is a non-empty set, $\leqslant$ is an ***order*** on P;

$\geqslant$ – the dual order of $\leqslant$

The bottom and the top element of a subset

P – ordered set; $H \subseteq P$;

$\top H$ – the greatest element (top element) of H, if it exists;

$\bot H$ – the smallest element (bottom element) of H, if it exists

Pointwise order of functions

$U \neq \emptyset$; P – ordered set; $f, g : U \to P$;

$f \leqslant g$ – $f(x) \leqslant g(x)$, $x \in U$

Ordered sets (cont.)

Definition:

P, Q – ordered sets; $f : P \to Q$ – function;

f – *isotone* or *order-preserving* if $x \leqslant y$ implies $f(x) \leqslant f(y)$, $x, y \in P$;

f – *antitone* or *order-inverting* if $x \leqslant y$ implies $f(x) \geqslant f(y)$, $x, y \in P$

Down-sets and up-sets

P – ordered set; $D, U \subseteq P$

D – a *down-set* if $x \in D$ and $y \in P$ such that $y \leqslant x$ implies $y \in D$;

$\emptyset \subseteq P$ – a down-set.

$\downarrow x = \{y \in P \,|\, y \leqslant x\}$ – the *principal down-set* generated by x

U – an *up-set* if $x \in U$ and $y \in P$ such that $y \geqslant x$ implies $y \in U$;

$\uparrow x = \{y \in P \,|\, y \geqslant x\}$ – the *principal up-set* generated by x

Ordered sets (cont.)

The directedness and completeness

$\emptyset \neq D \subseteq P$

D – an *upward directed* if for $a, b \in D$ there is $c \in D$ such that $a \leqslant c$ and $b \leqslant c$.

P – an *upward complete ordered set* if it has a least element and if every upward directed subset of P has a join.

A *downward directed* subset and a *downward complete ordered set* are defined dually.

Remark

D is an upward directed subset of P if and only if every finite subset of D has an upper bound in D.

The adjective ***complete*** – a different meaning applied to posets than applied to lattices.

Fixed points

Theorem 1. [Blyth(2005)]

P, Q – ordered sets; $f : P \to Q$ – a function;
The following conditions are equivalent:

(i) f is isotone;

(ii) the inverse image under f of every principal down-set of Q is a down-set of P;

(iii) the inverse image under f of every principal up-set of Q is an up-set of P.

Fixed points

P – an ordered set; $f : P \to P$ – an ***isotone function***

$a \in P$	– ***fixed point*** of f	$f(a) = a$
	– ***pre-fixed point*** of f	$f(a) \leqslant a$
	– ***post-fixed point*** of f	$a \leqslant f(a)$

Fixed points (cont.)

Notations

- $\mathtt{Fix}\,(f)$ – the set of all fixed points;
- $\mathtt{Post}\,(f)$ – the set of all post-fixed points;
- $\mathtt{Pre}\,(f)$ – the set of all pre-fixed points.

Knaster-Tarski fixed point theorem

P – a complete lattice; $f : P \to P$ – an isotone function;

***Post** (f) is a complete join-subsemilattice of P.*

***Pre** (f) is a complete meet-subsemilattice of P.*

***Fix** (f) is a complete lattice, $\top\mathbf{Fix}\,(f) = \top\mathbf{Post}\,(f)$ and $\bot\mathbf{Fix}\,(f) = \bot\mathbf{Pre}\,(f)$.*

Remark

The Knaster-Tarski fixed point theorem establishes existence of the least and the greatest fixed points, but it does not give an effective procedure for their computing.

Fixed points (cont.)

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- $\mathtt{Fix}\,(f)$ – the set of all fixed points;
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Fixed points (cont.)

The greatest fixed point

f – a meet-continuous function (which preserves all lower-directed meets);

Computing the least and greatest fixed points – based on the Kleene fixed point theorem.

The greatest fixed point of f can be "computed" as the meet of the descending Kleene chain of f defined as follows:

$$a_1 = 1, \quad a_{k+1} = f(a_k), \; k \in \mathbb{N}.$$

1 – the greatest element

Remark

Tha above equality enables either to effectively compute (if the sequence stabilizes at some a_k) or to approximate the greatest fixed point of f.

The greatest fixed point (cont.)

f – an isotone function (without meet-continuity);

$$\bar{a} \leqslant \bigwedge_{k\in\mathbb{N}} a_k,$$

$\bar{a}$ – the greatest fixed point of f

Examples

- In a finite lattice P the sequence $\{a_k\}_{k\in\mathbb{N}}$ must be finite, and its least element is equal to $\bar{a}$.
- the equality is provided in the above inequality when we deal with relations on a finite set or subsets of a finite set

Questions

Under what conditions the inverse image under $f : P \to Q$ (P,Q-ordered sets) of a principal down-set is also a principal down-set, and the inverse image under f of a principal up-set is also a principal up-set?

Residuated and residual functions

Theorem 2. [Blyth (2005)]

P, Q – ordered sets; $f : P \to Q$ – a function

The following conditions are equivalent:

(i) f is isotone and there exists an isotone function $g : Q \to P$ such that

$$I_P \leqslant f \circ g, \qquad g \circ f \leqslant I_Q;$$

(ii) there exists a function $g : Q \to P$ such that

$$f(x) \leqslant y \quad \Leftrightarrow \quad x \leqslant g(y), \quad x \in P,\ y \in Q;$$

(iii) the inverse image under f of every principal down-set of Q is a principal down-set of P;

(iv) f is isotone and the set $\{x \in P \mid f(x) \leqslant y\}$ has the greatest element, for every $y \in Q$.

Furthermore, if there is a function g which satisfies any of conditions (i) or (ii) it is unique.

Residuated and residual functions (cont.)

Residuated and residual functions

A function f that satisfies either of the equivalent conditions of the above theorem is called a ***residuated function***.

The unique function g that satisfies the condition (i) or (ii) of Theorem 2. is called the ***residual*** of f denoted by $f^\sharp$.

$$f^\sharp(y) = \top\{x \in P \mid f(x) \leqslant y\}, \quad y \in Q$$

Residuated and residual functions (cont.)

Theorem 3.

P, Q – ordered sets; $f : P \to Q$ – a function

The following conditions are equivalent:

(i) f is isotone and there exists an isotone function $g : Q \to P$ such that

$$I_Q \leqslant g \circ f, \qquad f \circ g \leqslant I_P;$$

(ii) there exists a function $g : Q \to P$ such that

$$f(x) \geqslant y \quad \Leftrightarrow \quad x \geqslant g(y), \quad x \in P, y \in Q$$

(iii) the inverse image under f of every principal up-set of Q is a principal up-set of P;

(iv) f is isotone and the set $\{x \in P \mid f(x) \geqslant y\}$ has the least element, for every $y \in Q$.

Furthermore, if there is a function g which satisfies any of conditions (i) or (ii) it is unique.

Residuated and residual functions (cont.)

Dually residuated (residual) functions

A function f that satisfies either of the equivalent conditions of Theorem 3. is called a ***dually residuated function***, or simpler a ***residual function***.

The unique function g that satisfies the condition (i) or (ii) of Theorem 3. is denoted by $f^\flat$.

$$f^\flat(y) = \bot\{x \in P \mid f(x) \geqslant y\}, \quad y \in Q.$$

Remark

If f is a residuated function, then $f^\sharp$ is a residual function and $(f^\sharp)^\flat = f$.

If f is a residual function, then $f^\flat$ is a residuated function and $(f^\flat)^\sharp = f$.

The composition of residuated functions is a residuated function.

The composition of residual functions is a residual function.

Residuated semigroups

Ordered semigroups

$(S,\otimes)$ – a semigroup; $(S,\leqslant)$ – an ordered set;

$(S,\otimes,\leqslant)$ – an ***ordered semigroup***

$$a \leqslant b \quad \Rightarrow \quad x\otimes a \leqslant x\otimes b \text{ and } a\otimes y \leqslant b\otimes y, \;\; a,b,x,y\in S$$

Translations on a semigroup $(S,\otimes)$

λ_a, $a\in S$ – the *left translation* on S determined by a defined by

$$\lambda_a(x) = a\otimes x, \quad x\in S$$

ϱ_a, $a\in S$ – the *right translation* on S determined by a defined by

$$\varrho_a(x) = x\otimes a, \quad x\in S$$

Residuated semigroups (cont.)

Residuated semigroups

$(S, \otimes, \leqslant)$ – an ordered semigroup;

S – ***right residuated*** if λ_a is a residuated function, for each $a \in S$;

$a \backslash b = \lambda_a^\sharp(b) = \top\{x \in S \mid a \otimes x \leqslant b\}, \ \ a, b \in S$ – the ***right residual*** of b by a;

S – ***left residuated*** if ϱ_a is a residuated function, for each $a \in S$;

$b/a = \varrho_a^\sharp(b) = \top\{x \in S \mid x \otimes a \leqslant b\}, \ \ a, b \in S$ – the ***left residual*** of b by a;

S – ***residuated semigroup*** if it is both right and left residuated;

$$a \otimes b \leqslant c \quad \Leftrightarrow \quad a \leqslant c/b \quad \Leftrightarrow \quad b \leqslant a \backslash c.$$

Equations and inequalities defined by residuated and residual functions

Inequalities defined by residuated and residual functions

P, Q – ordered sets; $c \in Q$ – an arbitrary fixed element;

x – an unknown taking values in P;

$f : P \to Q$ – a residuated function, $f(x) \leqslant c$ has the greatest solution $f^\sharp(c)$;

$f : P \to Q$ – a residual function, $c \leqslant f(x)$ has the least solution $f^\flat(c)$

Theorem 4. [Cuninghame-Green, Cechlárová (1995)]

An equation $f(x) = c$ is solvable if and only if $c = f(f^\sharp(c))$, and $f^\sharp(c)$ is also the greatest solution to this equation.

Theorem 5.

An equation $c = f(x)$ is solvable if and only if $c = f(f^\flat(c)$, and in this case $f^\flat(c)$ is the least solution to this equation.

$f(x) \leqslant g(x)$

Two-sided inequality

P, Q – complete lattices, $f, g : P \to Q$ – functions;

$$f(x) \leqslant g(x), \tag{1}$$

x is an unknown taking values in P

Theorem 6.

P, Q – complete lattices; $f, g : P \to Q$ – isotone functions

(a) If f is a residuated function, then the set of all solutions to inequality (1) is equal to $\mathtt{Post}\,(g \circ f^\sharp)$, and it is a complete join-subsemilattice of P. Consequently, inequality (1) has the greatest solution $\top\mathtt{Post}\,(g \circ f^\sharp)$.

(b) If g is a residual function, then the set of all solutions to inequality (1) is equal to $\mathtt{Pre}\,(f \circ g^\flat)$, and it is a complete meet-subsemilattice of P. Consequently, inequality (1) has the least solution $\bot\mathtt{Pre}\,(f \circ g^\flat)$.

$f(x) \leqslant g(x)$ (cont.)

Corollary 1.

P, Q – complete lattices; $f, g : P \to Q$ – f is a residuated and g is a residual function

The set of all solutions to inequality $f(x) \leqslant g(x)$ is equal to $\mathtt{Post}\,(g \circ f^\sharp) = \mathtt{Pre}\,(f \circ g^\flat)$, and it is a complete sublattice of P.

Consequently, inequality $f(x) \leqslant g(x)$ has the greatest solution $\top\mathtt{Post}\,(g \circ f^\sharp)$ and the least solution $\bot\mathtt{Pre}\,(f \circ g^\flat)$.

$f(x) = g(x)$

Two-sided equation

P, Q – complete lattices, $f, g : P \to Q$ – functions

$$f(x) = g(x), \tag{2}$$

x is an unknown taking values in P

Theorem 7.

P, Q – complete lattices; $f, g : P \to Q$ – isotone functions

(a) If f and g are residuated functions, then the set of all solutions to equation (2) is equal to $\mathtt{Post}\,(g \circ f^\sharp \wedge f \circ g^\sharp)$, and it is a complete join-subsemilattice of P, so inequality (1) has the greatest solution $\top\mathtt{Post}\,(g \circ f^\sharp \wedge f \circ g^\sharp)$.

(b) If f and g are residual functions, then then the set of all solutions to equation (2) is equal to $\mathtt{Pre}\,(f \circ g^\flat \wedge g \circ f^\flat)$, and it is a complete meet-subsemilattice of P, so inequality (1) has the least solution $\bot\mathtt{Pre}\,(f \circ g^\flat \wedge g \circ f^\flat)$.

$f(x) \leqslant x;\ x \leqslant f(x);\ f(x) = x$

Special cases of two-sided inequalities and equations

P – a complete lattice, $f : P \to P$ – a function

$$f(x) \leqslant x, \tag{3}$$

$$x \leqslant f(x), \tag{4}$$

$$f(x) = x, \tag{5}$$

x is an unknown taking values in P

Remarks

f – an isotone function;

The sets of solutions to (3), (4) and (5), as well as the corresponding greatest and least solutions – completely characterized by the Knaster-Tarski fixed point theorem.

As the identity function is both residuated and residual, then the following results are obtained as direct consequences of Theorems 6. and 7.

$f(x) \leqslant x$; $x \leqslant f(x)$; $f(x) = x$ (cont.)

Corollary 2.

P, Q – complete lattices; $f : P \to Q$ – a residuated function

(a) The set of all solutions to inequality $f(x) \leqslant x$ is equal to $\mathtt{Post}\,(f^\sharp) = \mathtt{Pre}\,(f)$, and it is a complete sublattice of P.
Consequently, inequality $f(x) \leqslant x$ has the greatest solution $\top\mathtt{Post}\,(f^\sharp)$ and the least solution $\bot\mathtt{Pre}\,(f)$.

(b) The set of all solutions to equation $f(x) = x$ is equal to $\mathtt{Post}\,(f \wedge f^\sharp)$, and it is a complete join-subsemilattice of P.
Consequently, equation $f(x) = x$ has the greatest solution $\top\mathtt{Post}\,(f \wedge f^\sharp)$.

$f(x) \leqslant x$; $x \leqslant f(x)$; $f(x) = x$ (cont.)

Corollary 3.

P, Q – complete lattices; $f : P \to Q$ – a residual function;

(a) The set of all solutions to inequality $x \leqslant f(x)$ is equal to $\mathtt{Post}\,(f) = \mathtt{Pre}\,(f^\flat)$, and it is a complete sublattice of P.
Consequently, inequality $x \leqslant f(x)$ has the greatest solution $\top\mathtt{Post}\,(f)$ and the least solution $\bot\mathtt{Pre}\,(f^\flat)$.

(b) The set of all solutions to equation $f(x) = x$ is equal to $\mathtt{Pre}\,(f \wedge f^\flat)$, and it is a complete meet-subsemilattice of P.
Consequently, equation $f(x) = x$ has the least solution $\bot\mathtt{Pre}\,(f \wedge f^\flat)$.

f(x)= g(y)

Two-sided equality with two unknowns

P, Q, R – complete lattices; $f : P \to R$, $g : Q \to R$ – functions

$$f(x) = g(y), \tag{6}$$

unknowns x and y taking values in P and Q

solution – any pair $(a, b) \in P \times Q$ such that $f(a) = g(b)$

Theorem 8.

P, Q, R – complete lattices; $f : P \to R$, $g : Q \to R$ – residuated functions

The equation $f(x) = g(y)$ is equivalent to inequality

$$z \leqslant \phi(z),$$

where z is an unknown taking values in $P \times Q$ and $\phi : P \times Q \to R \times R$ is an isotone function defined by

$$\phi(a, b) = \big((g \circ f^\sharp)(b), (f \circ g^\sharp)(a)\big), \;\; a \in P, b \in Q.$$

Consequently, the set of all solutions to equation $f(x) = g(y)$ is equal to $\mathtt{Post}\,(\phi)$, and $\top\mathtt{Post}\,(\phi)$ is the greatest solution to $f(x) = g(y)$.

$f(x) = g(y)$ (cont.)

Theorem 9.

P, Q, R – complete lattices; $f : P \to R$, $g : Q \to R$ – residual functions

The equation $f(x) = g(y)$ is equivalent to

$$\psi(z) \leqslant z,$$

where z is an unknown taking values in $P \times Q$ and $\psi : P \times Q \to R \times R$ is an isotone function defined by

$$\psi(a, b) = \Big((g \circ f^{\flat})(b), (f \circ g^{\flat})(a)\Big), \quad a \in P, b \in Q.$$

Consequently, the set of all solutions to equation $f(x) = g(y)$ is equal to $\texttt{Pre}\,(\psi)$, and $\perp\texttt{Pre}\,(\psi)$ is the least solution to $f(x) = g(y)$.

$f(x) = g(y)$ (cont.)

Theorem 10.

P, Q, R – complete lattices; $f : P \to R$, $g : Q \to R$ – arbitrary functions

The equation $f(x) = g(y)$ is equivalent to

$$\alpha(z) = \beta(z),$$

where z is an unknown taking values in $P \times Q$ and $\alpha, \beta : P \times Q \to R \times R$ are isotone functions defined by

$$\alpha(a,b) = ((f(a), g(b)), \qquad \beta(a,b) = (g(b), f(a)), \;\; a \in P, b \in Q.$$

If f and g are residuated, resp. residual functions, then α and β are also residuated, resp. residual functions.

Residuals of a fuzzy relation by a fuzzy relation

Residuals of a fuzzy relation by a fuzzy relation

U, V, W – non-empty sets; $A \in \mathcal{R}(U,V)$, $B \in \mathcal{R}(V,W)$ – given fuzzy relations;

$\lambda_A : \mathcal{R}(V,W) \to \mathcal{R}(U,W)$ defined by $\lambda_A(X) = A \circ X$ – a residuated function;

$\lambda_A^\sharp : \mathcal{R}(U,W) \to \mathcal{R}(V,W)$ given by $\lambda_A^\sharp(Y) = A \backslash Y$ – the ***right residual*** of Y by A;

$$(A\backslash Y)(v,w) = \bigwedge_{u\in U} A(u,v) \to Y(u,w), \quad v \in V, w \in W.$$

$\varrho_B : \mathcal{R}(U,V) \to \mathcal{R}(U,W)$ defined by $\varrho_B(X) = X \circ B$ – a residuated function;

$\varrho_B^\sharp : \mathcal{R}(U,W) \to \mathcal{R}(U,V)$ given by $\varrho_B^\sharp(Y) = Y/B$ – the ***left residual*** of Y by B;

$$(Y/B)(u,v) = \bigwedge_{w\in W} B(v,w) \to Y(u,w), \quad u \in U, v \in V.$$

Equatuions and inequalities with unknown fuzzy relations

Remark

$\iota : X \mapsto X$ and $\tau : X \mapsto X^{-1}$ – both residuated and residual functions;

Notation

A, B, C, etc. – given fuzzy relations;

X, Y, etc. – unknown fuzzy relations;

$\bowtie$ – a joker sign which replaces anyone of the signs $\leqslant$, $\geqslant$ or $=$.

Equations and inequalities with given and unknown fuzzy relations, defined by residuated functions

	Equation/inequality	Given fuzzy relations	Unknown fuzzy relations	Written by residuated functions
1.	$A \circ X \bowtie B$	$A \in \mathcal{R}(U,V)$, $B \in \mathcal{R}(U,W)$	$X :\in \mathcal{R}(V,W)$	$\lambda_A(X) \bowtie B$
2.	$X \circ A \bowtie B$	$A \in \mathcal{R}(V,W)$, $B \in \mathcal{R}(U,W)$	$X \in \mathcal{R}(U,V)$	$\varrho_A(X) \bowtie B$
3.	$A \circ X \bowtie X$	$A \in \mathcal{R}(U,U)$	$X \in \mathcal{R}(U,V)$	$\lambda_A(X) \bowtie \iota(X)$
4.	$X \circ A \bowtie X$	$A \in \mathcal{R}(V,V)$	$X \in \mathcal{R}(U,V)$	$\varrho_A(X) \bowtie \iota(X)$
5.	$A \circ X \bowtie X \circ A$	$A \in \mathcal{R}(U,U)$	$X \in \mathcal{R}(U,U)$	$\lambda_A(X) \bowtie \varrho_A(X)$
6.	$A \circ X^{-1} \bowtie X^{-1} \circ A$	$A \in \mathcal{R}(U,U)$	$X \in \mathcal{R}(U,U)$	$\tau \circ \lambda_A(X) \bowtie \tau \circ \varrho_A(X)$
7.	$A \circ X \bowtie X \circ B$	$A, B \in \mathcal{R}(U,U)$	$X \in \mathcal{R}(U,U)$	$\lambda_A(X) \bowtie \varrho_B(X)$
8.	$A \circ X^{-1} \bowtie X^{-1} \circ B$	$A, B \in \mathcal{R}(U,U)$	$X \in \mathcal{R}(U,U)$	$\tau \circ \lambda_A(X) \bowtie \tau \circ \varrho_B(X)$
9.	$A \circ X \bowtie X \circ B$	$A \in \mathcal{R}(U,U)$, $B \in \mathcal{R}(V,V)$	$X \in \mathcal{R}(U,V)$	$\lambda_A(X) \bowtie \varrho_B(X)$
10.	$X^{-1} \circ A \bowtie B \circ X^{-1}$	$A \in \mathcal{R}(U,U)$, $B \in \mathcal{R}(V,V)$	$X \in \mathcal{R}(U,V)$	$\tau \circ \varrho_A(X) \bowtie \tau \circ \lambda_B(X)$
11.	$A \circ X \bowtie B \circ X$	$A, B \in \mathcal{R}(U,V)$	$X \in \mathcal{R}(V,W)$	$\lambda_A(X) \bowtie \lambda_B(X)$
12.	$X \circ A \bowtie X \circ B$	$A, B \in \mathcal{R}(V,W)$	$X \in \mathcal{R}(U,V)$	$\varrho_A(X) \bowtie \varrho_B(X)$
13.	$A \circ X = A \circ Y$	$A \in \mathcal{R}(U,V)$	$X, Y \in \mathcal{R}(V,W)$	$\lambda_A(X) = \lambda_A(Y)$
14.	$X \circ A = Y \circ A$	$A \in \mathcal{R}(V,W)$	$X, Y \in \mathcal{R}(U,V)$	$\varrho_A(X) = \varrho_A(Y)$
15.	$X \circ A = A \circ Y$	$A \in \mathcal{R}(U,V)$	$X \in \mathcal{R}(U,U)$, $Y \in \mathcal{R}(V,V)$	$\varrho_A(X) = \lambda_A(Y)$
16.	$A \circ X = B \circ Y$	$A \in \mathcal{R}(U,V)$, $B \in \mathcal{R}(U,W)$	$X \in \mathcal{R}(V,Z)$, $Y \in \mathcal{R}(W,Z)$	$\lambda_A(X) = \lambda_B(Y)$
17.	$X \circ A = Y \circ B$	$A \in \mathcal{R}(V,Z)$, $B \in \mathcal{R}(W,Z)$	$X \in \mathcal{R}(U,V)$, $Y \in \mathcal{R}(U,W)$	$\varrho_A(X) = \varrho_B(Y)$
18.	$X \circ A = B \circ Y$	$A \in \mathcal{R}(V,Z)$, $B \in \mathcal{R}(U,W)$	$X \in \mathcal{R}(U,V)$, $Y \in \mathcal{R}(W,Z)$	$\varrho_A(X) = \lambda_B(Y)$

Equatuions and inequalities with unknown fuzzy relations (cont.)

Classification of equations and inequalities with given and unknown fuzzy relations:

1. and 2. – studied by Sanchez (1974,1976,1978);

Cuninghame-Green and Cechlárová (1995) – a more general form through residuated functions

5. and 6. – homogeneous weakly linear inequalities and equations [Ignjatović, Ćirić, Bogdanović (2010)]

9. and 10. – heterogeneous weakly linear inequalities and equations [Ignjatović, Ćirić, Damljanović, Jančić (2011)]

15. – studied by Stanković, Ignjatović, Ćirić (2011), in the crisp case

Applications

Equations of the form 15 can be applied to the analysis of data represented by Boolean and fuzzy data tables (analysis of two-mode social networks, i.e. of actor-event social networks represented by bipartite graphs).

Residuals of a fuzzy set by a fuzzy relation

Residuals of a fuzzy set by a fuzzy relation

U, V – non-empty sets; $A \in \mathcal{R}(U,V)$ – a given fuzzy relation

$\varrho_A : \mathcal{F}(U) \to \mathcal{F}(V)$ defined by $\varrho_A(\mathbf{x}) = \mathbf{x} \circ A$ – a residuated function, where

$$(\mathbf{x} \circ A)(v) = \bigvee_{u \in U} \mathbf{x}(u) \otimes A(u,v), \quad \mathbf{x} \in \mathcal{F}(U), v \in V$$

$\varrho_A^\sharp : \mathcal{F}(V) \to \mathcal{F}(U)$ defined by $\varrho_A^\sharp(\mathbf{y}) = \mathbf{y}/A$ – a residual function, where

$$(\mathbf{y}/A)(u) = \bigwedge_{v \in V} A(u,v) \to \mathbf{y}(v), \quad \mathbf{y} \in \mathcal{F}(V), u \in U$$

$\lambda_A : \mathcal{F}(V) \to \mathcal{F}(U)$ defined by $\lambda_A(\mathbf{y}) = A \circ \mathbf{y}$ – a residuated function, where

$$(A \circ \mathbf{y})(u) = \bigvee_{v \in V} A(u,v) \otimes \mathbf{y}(v), \quad \mathbf{y} \in \mathcal{F}(V), u \in U$$

$\lambda_A^\sharp : \mathcal{F}(U) \to \mathcal{F}(V)$ defined by $\lambda_A^\sharp(\mathbf{x}) = A \backslash \mathbf{x}$ – a residual function, where

$$(A \backslash \mathbf{x})(v) = \bigwedge_{u \in U} A(u,v) \to \mathbf{x}(u), \quad \mathbf{x} \in \mathcal{F}(U), v \in V$$

Equations and inequalities with given fuzzy relations/sets and unknown fuzzy sets, defined by residuated functions

	Equation/inequality	Given fuzzy relations/sets	Unknown fuzzy relations	Written by residuated functions
1.	$A \circ \mathbf{x} \bowtie \mathbf{b}$	$A \in \mathcal{R}(U, V)$, $\mathbf{b} \in \mathcal{F}(U)$	$\mathbf{x} \in \mathcal{F}(V)$	$\lambda_A(\mathbf{x}) \bowtie \mathbf{b}$
2.	$\mathbf{x} \circ A \bowtie \mathbf{b}$	$A \in \mathcal{R}(U, V)$, $\mathbf{b} \in \mathcal{F}(V)$	$\mathbf{x} \in \mathcal{F}(U)$	$\varrho_A(\mathbf{x}) \bowtie \mathbf{b}$
3.	$A \circ \mathbf{x} \bowtie \mathbf{x}$	$A \in \mathcal{R}(U, U)$	$\mathbf{x} \in \mathcal{F}(U)$	$\lambda_A(\mathbf{x}) \bowtie \iota(\mathbf{x})$
4.	$\mathbf{x} \circ A \bowtie \mathbf{x}$	$A \in \mathcal{R}(U, U)$	$\mathbf{x} \in \mathcal{F}(U)$	$\varrho_A(\mathbf{x}) \bowtie \iota(\mathbf{x})$
5.	$A \circ \mathbf{x} \bowtie B \circ \mathbf{x}$	$A, B \in \mathcal{R}(U, V)$	$\mathbf{x} \in \mathcal{F}(V)$	$\lambda_A(\mathbf{x}) \bowtie \lambda_B(\mathbf{x})$
6.	$\mathbf{x} \circ A \bowtie \mathbf{x} \circ B$	$A, B \in \mathcal{R}(U, V)$	$\mathbf{x} \in \mathcal{F}(U)$	$\varrho_A(\mathbf{x}) \bowtie \varrho_B(\mathbf{x})$
7.	$\mathbf{x} \circ A \bowtie B \circ \mathbf{x}$	$A \in \mathcal{R}(U, V)$, $B \in \mathcal{R}(V, U)$	$\mathbf{x} \in \mathcal{F}(U)$	$\varrho_A(\mathbf{x}) \bowtie \lambda_B(\mathbf{x})$
8.	$\mathbf{x} \circ A \bowtie A \circ \mathbf{x}$	$A \in \mathcal{R}(U, U)$	$\mathbf{x} \in \mathcal{F}(U)$	$\varrho_A(\mathbf{x}) \bowtie \lambda_A(\mathbf{x})$
9.	$\mathbf{x} \circ A = \mathbf{y} \circ A$	$A \in \mathcal{R}(U, V)$	$\mathbf{x}, \mathbf{y} \in \mathcal{F}(U)$	$\varrho_A(\mathbf{x}) = \varrho_A(\mathbf{y})$
10.	$A \circ \mathbf{x} = A \circ \mathbf{y}$	$A \in \mathcal{R}(U, V)$	$\mathbf{x}, \mathbf{y} \in \mathcal{F}(V)$	$\lambda_A(\mathbf{x}) = \lambda_A(\mathbf{y})$
11.	$\mathbf{x} \circ A = A \circ \mathbf{y}$	$A \in \mathcal{R}(U, U)$	$\mathbf{x}, \mathbf{y} \in \mathcal{F}(U)$	$\varrho_A(\mathbf{x}) = \lambda_A(\mathbf{y})$
12.	$\mathbf{x} \circ A = \mathbf{y} \circ B$	$A \in \mathcal{R}(U, W)$, $B \in \mathcal{R}(V, W)$	$\mathbf{x} \in \mathcal{F}(U)$, $\mathbf{y} \in \mathcal{F}(V)$	$\varrho_A(\mathbf{x}) = \varrho_B(\mathbf{y})$
13.	$A \circ \mathbf{x} = B \circ \mathbf{y}$	$A \in \mathcal{R}(U, V)$, $B \in \mathcal{R}(U, W)$	$\mathbf{x} \in \mathcal{F}(V)$, $\mathbf{y} \in \mathcal{F}(W)$	$\lambda_A(\mathbf{x}) = \lambda_B(\mathbf{y})$
14.	$\mathbf{x} \circ A = B \circ \mathbf{y}$	$A \in \mathcal{R}(U, V)$, $B \in \mathcal{R}(V, W)$	$\mathbf{x} \in \mathcal{F}(U)$, $\mathbf{y} \in \mathcal{F}(W)$	$\varrho_A(\mathbf{x}) = \lambda_B(\mathbf{y})$

Classification of equations and inequalities with given fuzzy relations and unknown fuzzy sets

1. and 2. – studied by Sanchez (1974,1976,1978) along with the corresponding equations and inequalities with fuzzy relations;

Cuninghame-Green and Cechlárová (1995) – a more general form through residuated functions

3. and 4. – [Sanchez (1978,1981), Ćirić, Ignjatović, Šešelja, Tepavčević (2011)]

5. – fuzzy bilinear equations [Tang (1988), Li (1992), Zhang (1995)]

Residuals of a fuzzy set by a fuzzy set

Residuals of a fuzzy set by a fuzzy set

U, V – non-empty sets; $\mathbf{a} \in \mathcal{F}(U)$, $\mathbf{b} \in \mathcal{F}(V)$ – given fuzzy sets;

$\lambda_{\mathbf{a}} : \mathcal{R}(U,V) \to \mathcal{F}(V)$ defined by $\lambda_{\mathbf{a}}(X) = \mathbf{a} \circ X$, $X \in \mathcal{R}(U,V)$ – a residuated function;

$\lambda_{\mathbf{a}}^{\sharp} : \mathcal{F}(V) \to \mathcal{R}(U,V)$ given by $\lambda_{\mathbf{a}}^{\sharp}(\mathbf{y}) = \mathbf{a}\backslash\mathbf{y}$, $\mathbf{y} \in \mathcal{F}(V)$ – the residual, where

$$(\mathbf{a}\backslash\mathbf{y})(u,v) = \mathbf{a}(u) \to \mathbf{y}(v), \;\; \mathbf{a}\backslash\mathbf{y} \in \mathcal{R}(U,V), \; (u,v) \in U \times V$$

$\varrho_{\mathbf{b}} : \mathcal{R}(U,V) \to \mathcal{F}(U)$ defined by $\varrho_{\mathbf{b}}(X) = X \circ \mathbf{b}$, $X \in \mathcal{R}(U,V)$ – a residuated function;

$\varrho_{\mathbf{b}}^{\sharp} : \mathcal{F}(U) \to \mathcal{R}(U,V)$ given by $\varrho_{\mathbf{b}}^{\sharp}(\mathbf{x}) = \mathbf{x}/\mathbf{b}$, $\mathbf{x} \in \mathcal{F}(U)$ – the residual, where

$$(\mathbf{x}/\mathbf{b})(u,v) = \mathbf{b}(v) \to \mathbf{x}(u), \;\; \mathbf{x}/\mathbf{b} \in \mathcal{R}(U,V), \; (u,v) \in U \times V$$

Equations and inequalities with given fuzzy sets and unknown fuzzy relations, defined by residuated functions

	Equation/inequality	Given fuzzy sets	Unknown fuzzy relations	Written by residuated functions
1.	$\mathbf{a} \circ X \bowtie \mathbf{b}$	$\mathbf{a} \in \mathcal{F}(U)$, $\mathbf{b} \in \mathcal{F}(V)$	$X \in \mathcal{R}(U,V)$	$\lambda_{\mathbf{a}}(X) \bowtie \mathbf{b}$
2.	$X \circ \mathbf{b} \bowtie \mathbf{a}$	$\mathbf{a} \in \mathcal{F}(U)$, $\mathbf{b} \in \mathcal{F}(V)$	$X \in \mathcal{R}(U,V)$	$\varrho_{\mathbf{b}}(X) \bowtie \mathbf{a}$
3.	$\mathbf{a} \circ X \bowtie \mathbf{a}$	$\mathbf{a} \in \mathcal{F}(U)$	$X \in \mathcal{R}(U,U)$	$\lambda_{\mathbf{a}}(X) \bowtie \mathbf{a}$
4.	$X \circ \mathbf{a} \bowtie \mathbf{a}$	$\mathbf{a} \in \mathcal{F}(U)$	$X \in \mathcal{R}(U,U)$	$\varrho_{\mathbf{a}}(X) \bowtie \mathbf{a}$
5.	$\mathbf{a} \circ X \bowtie \mathbf{b} \circ X$	$\mathbf{a}, \mathbf{b} \in \mathcal{F}(U)$	$X \in \mathcal{R}(U,V)$	$\lambda_{\mathbf{a}}(X) \bowtie \lambda_{\mathbf{b}}(X)$
6.	$X \circ \mathbf{a} \bowtie X \circ \mathbf{b}$	$\mathbf{a}, \mathbf{b} \in \mathcal{F}(V)$	$X \in \mathcal{R}(U,V)$	$\varrho_{\mathbf{a}}(X) \bowtie \varrho_{\mathbf{b}}(X)$
7.	$\mathbf{a} \circ X \bowtie X \circ \mathbf{b}$	$\mathbf{a}, \mathbf{b} \in \mathcal{F}(U)$	$X \in \mathcal{R}(U,U)$	$\lambda_{\mathbf{a}}(X) \bowtie \varrho_{\mathbf{b}}(X)$
8.	$\mathbf{a} \circ X = \mathbf{a} \circ Y$	$\mathbf{a} \in \mathcal{F}(U)$	$X, Y \in \mathcal{R}(U,V)$	$\lambda_{\mathbf{a}}(X) = \lambda_{\mathbf{a}}(Y)$
9.	$X \circ \mathbf{a} = Y \circ \mathbf{a}$	$\mathbf{a} \in \mathcal{F}(V)$	$X, Y \in \mathcal{R}(U,V)$	$\varrho_{\mathbf{a}}(X) = \varrho_{\mathbf{a}}(Y)$
10.	$\mathbf{a} \circ X = Y \circ \mathbf{a}$	$\mathbf{a} \in \mathcal{F}(U)$	$X \in \mathcal{R}(U,V)$, $Y \in \mathcal{R}(V,U)$	$\lambda_{\mathbf{a}}(X) = \varrho_{\mathbf{a}}(Y)$
11.	$\mathbf{a} \circ X = \mathbf{b} \circ Y$	$\mathbf{a} \in \mathcal{F}(U)$, $\mathbf{b} \in \mathcal{F}(V)$	$X \in \mathcal{R}(U,W)$, $Y \in \mathcal{R}(V,W)$	$\lambda_{\mathbf{a}}(X) = \lambda_{\mathbf{b}}(Y)$
12.	$X \circ \mathbf{a} = Y \circ \mathbf{b}$	$\mathbf{a} \in \mathcal{F}(V)$, $\mathbf{b} \in \mathcal{F}(W)$	$X \in \mathcal{R}(U,V)$, $Y \in \mathcal{R}(U,W)$	$\varrho_{\mathbf{a}}(X) = \varrho_{\mathbf{b}}(Y)$
13.	$\mathbf{a} \circ X = Y \circ \mathbf{b}$	$\mathbf{a} \in \mathcal{F}(U)$, $\mathbf{b} \in \mathcal{F}(W)$	$X \in \mathcal{R}(U,V)$, $Y \in \mathcal{R}(V,W)$	$\lambda_{\mathbf{a}}(X) = \varrho_{\mathbf{b}}(Y)$

Moore-Penrose equations

Moore-Penrose equations

U, V – nonempty sets; $A \in \mathcal{R}(U, V)$ – a given fuzzy relation;

$X \in \mathcal{R}(U, V)$ – an unknown fuzzy relation

(G1) $A \circ X \circ A = A$;

(G2) $X \circ A \circ X = X$;

(G3) $(A \circ X)^{-1} = A \circ X$;

(G4) $(X \circ A)^{-1} = X \circ A$

Moore-Penrose inverse

System (G1) - (G4) has the unique solution – the ***Moore-Penrose inverse*** **of** A

Remark:

The equation (G2) is independent, but the left side of (G1) is the residuated function, and in (G3) and (G4) residuated functions are on both sides, so the equations (G1), (G3) and (G4) can be resolved by using above mention methods.

THANK YOU FOR YOUR ATTENTION!