Pre pseudo-effect algebras

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Outline

- Motivation - results of I. Chajda, J. Kühr
- Pre pseudo-effect algebras - several definitions
- Basic results
- Gallery of structures
- Complexity of the search
- Future research - open problems

Motivation

- Each lattice ordered orthoalgebra is OML.
- I. Chajda, J. Kühr - to find a structure such that ortholattices (without orthomodular law) are represented in the similar way.
- pre effect algebras, pre orthoalgebras
- pre effect algebra = effect algebra - unique existence of orthosupplement
- i.e. orthosupplement need no longer be the only element such that
- $a + b = 1$
- Pre orthoalgebras = pre effect algebra + property $\exists a + a$, iff $a = 0$.

Definitions

Pseudo effect-algebra $= (A; \oplus, {}^L, {}^R, 0, 1)$ such that

- $a \oplus b$, $(a \oplus b) \oplus c$ exist, iff $b \oplus c$, $a \oplus (b \oplus c)$ exist and in such case $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
- for any $a \in A$ there are unique elements e, f such that $a + e = f + a = 1$. $(e := a^R)$, $(f := a^L)$;
- if $a + b$ is defined then there are elements c, d such that $a + b = c + a = b + d$;
- $1 + a$, $a + 1$ are defined then $a = 0$.

Order: $a \leq b$, iff $\exists c\ a + c = b$, iff $\exists d\ d + a = b$.

Weak pre-pseudo effect algebras

Weak pre-effect-algebra $= (A; \oplus, ^L, ^R, 0, 1)$ such that

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- for any $a \in A$ $a + a^R = a^L + a = 1$;
- relation $a \leq b$, iff $a \oplus b^R$ is defined, iff $b^L + a$ is defined is a partial order;
- $1 + a$, $a + 1$ are defined then $a = 0$.
- 1 and 0 are comparable to all elements

Order left: $a \sqsubseteq_L b$, iff $\exists c \; a + c = b$
Order right: $a \sqsubseteq_R b$, iff $\exists d \; d + a = b$.
Order $\sqsubseteq$: if it is both left and right order

Pre pseudo-effect algebras

Pre-effect-algebra $= (A; \oplus, ^L, ^R, 0, 1)$ such that

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- relation $a \leq b$, iff $a \oplus b^R$ is defined, iff $b^L + a$ is defined is a partial order;
- if $a + b$ is defined then there are elements c, d such that $a + b = c + a = b + d$;
- $1 + a$, $a + 1$ are defined then $a = 0$;
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Order $\sqsubseteq$: $a \sqsubseteq b$, iff $\exists c$ $a + c = b$, iff $\exists d$ $d + a = b$.

(Weak) Pre pseudo orthoalgebras

(Weak) Pre pseudo orthoalgebras $=$ (weak) pre pseudo effect algebras $+$ property $a + a$ is defined, then $a = 0$.

Results

Let A be a weak pre pseudo-effect algebra. Then

- $a+0$, $0+a$ are defined and $a+0=0+a=a$;
- 0 is the bottom and 1 is the top elements in $(A,\leq)$;
- $a=a^{RL}=a^{LR}$;
- if $a+b$ is defined, then $a\leq a+b$;
- if $b+a$ is defined, then $a\leq b+a$;
- if $a+b$ is defined and $a+b=a$, then $b=0$;
- if $b+a$ is defined and $b+a=a$, then $b=0$;
- $b\leq c$, then if $a+c$ is defined then $a+b$ is defined and $a+b\leq a+c$
- $b\leq c$, then if $c+a$ is defined then $b+a$ is defined and $b+a\leq c+a$
- $a+b=0$, then $a=b=0$

It need not be cancelative, i.e. $a+b=a+c$ (, resp. $b+a=c+a$) , then $b=c$.

Results contd.

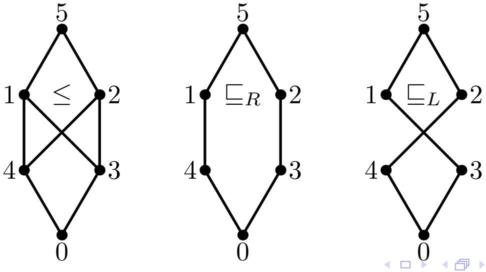
- if WprePEA is commutative, then it is also prePEA
- (prePEA) if $a \sqsubseteq b$, then $a \leq b$
- (WprePEA) if $a \sqsubseteq_L b$, or $a \sqsubseteq_R b$, then $a \leq b$
- (prePEA) if $\leq = \sqsubseteq$, then it is PEA

Gallery of structures

The smallest example of weak pre-pseudoeffect algebra that is not pre-preudoeffect algebra

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**
0	0	1	2	3	4	5
1	1	-	-	5	-	-
2	2	-	-	-	5	-
3	3	-	5	5	1	-
4	4	5	-	2	5	-
5	5	-	-	-	-	-

	0	**1**	**2**	**3**	**4**	**5**
L	5	4	3	1	2	0
R	5	3	4	2	1	0



Gallery of structures

The smallest example of Pre-pseudoeffect algebra that is not pseudoeffect algebra

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**
0	0	1	2	3	4	5
1	1	-	-	5	-	-
2	2	-	-	-	5	-
3	3	-	5	5	5	-
4	4	5	-	5	5	-
5	5	-	-	-	-	-

	0	**1**	**2**	**3**	**4**	**5**
L	5	4	3	1	2	0
R	5	3	4	2	1	0

Gallery of structures

The smallest pseudoeffect algebra that is not effect algebra.

$\oplus$	**0**	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	-	4	-	-
2	2	-	-	4	-
3	3	4	-	-	-
4	4	-	-	-	-

	0	**1**	**2**	**3**	**4**
L	4	3	1	2	0
R	4	2	3	1	0

Gallery of results

There are **36** 5-element pre pseudoeffect algebras that reduce to **8** distict algebras from which just **one** is non-commutative.

There are **648** 6-element pre pseudoeffect algebras that reduce to **38** distinct algebras for which **3** are non-commutative.

	2	**3**	**4**	**5**	**6**
total	1	1	6	36	648
nonIso	1	1	4	8	38
nonComm	0	0	0	1	3

Complexity of the search

- For $\oplus$: $(n-2)\times(n-2)$ elements and each is taken from $\{-, 0, 1, \ldots, n-1\}$, rough estimate $(n+1)^{(n-2)^2}$.
- left and right orthosupplement $(n-2)!$
- For each possibility, we need to test
-
    - associativity, $O(n^3)$
    - existence of left and right orthosupplement, $O(n^2)$
    - for PrePEA - $\exists a+b$, then there are c, d such that $a+b=c+a=b+d$, $O(n^2)$
    - partial order $a \leq b$, iff $\exists a \oplus b^R$, iff $\exists b^L \oplus a$: antisymmetry $O(n^2)$, transitivity $O(n^3)$
- Total (worst case): $(n+1)^{(n-2)^2}(O((n-2)!)^2).O(n^3)$

Complexity - Improvements

- Since $a = a^{RL} = a^{LR}$ then if we know one orthosupplement then in $O(n)$ we are able to compute the other one
- Thus $((n-2)!)^2$ is reduced to $(n-2)!O(n)$

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- Similarly, $a + b = a$, iff $b = 0$ and $a + b = b$, iff $a = 0$, thus $a + b \neq a, b$
- $n^{(n-2)^2}$ is reduced to $(n-2)^{(n-2)(n-3)}.(n-1)^{n-2}$

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- Total (worst c.): $(n-2)^{(n-2)(n-3)} \cdot (n-1)^{n-2} \cdot O((n-2)!) \cdot O(n^4)$
- E.g. for n = 6, the computation is ≈ 3756 times faster

Complexity - Improvements

Since for each element there are its left and right complements, each column (row) needs to contain at least one unit. This reduces number of possibilities per column to the value:

$$(n-1)[(n-3)^{n-4} + (n-3)^{n-5}(n-2) + \dots \\ + (n-3)(n-2)^{n-5} + (n-2)^{n-4}] + (n-3)^{n-3}.$$

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For $n = 6$ it is ≈ 5.19 times faster than previous method

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  For $n = 6$ it is mere $\approx 2.96 \cdot 10^{11}$, i.e. it is necessary $2.33 \cdot 10^{10}$ more time.
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- Prob1: Introduce noncommutative version of generalized pre effect algebras (generalized pre D-posets).
- Prob2: T. Vetterlein introduced so called weak pseudo-effect algebras. Is there a relation to ChKu's strong pre effect algebras?

References

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