Triangular norms based submeasures

Ondrej Hutník

Institute of Mathematics
Faculty of Science
Pavol Jozef Šafárik University in Košice
Jesenná 5, 040 01 Košice
Slovakia

E-mail: ondrej.hutnik@upjs.sk

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A brief history and motivation

- Let Σ be a ring of subsets of a fixed (non-empty) set Ω. A mapping $\eta : \Sigma \to \overline{\mathbb{R}}_+$ such that
  (i) $\eta(\emptyset) = 0$;
  (ii) $\eta(E) \leq \eta(F)$ for $E, F \in \Sigma$ such that $E \subset F$;
  (iii) $\eta(E \cup F) \leq \eta(E) + \eta(F)$ whenever $E, F \in \Sigma$;

  is said to be a numerical submeasure.
- MENGER'S IDEA OF PROBABILISTIC METRIC SPACES: let Ω be a non-empty set, $\mathcal{F} : \Omega \times \Omega \to \Delta^+$ a function which assigns to each pair $(p, q) \in \Omega \times \Omega$ a distance distribution function $F_{p,q} \in \Delta^+$, and $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ a triangle function. The triple $(\Omega, \mathcal{F}, \tau)$ is called a probabilistic metric space (PM-space) if the following properties hold for all $p, q, r \in \Omega$:
  (1) $F_{p,q} = \varepsilon_0$ if and only if $p = q$;
  (2) $F_{p,q} = F_{q,p}$;
  (3) $F_{p,r} \geq \tau(F_{p,q}, F_{q,r})$.
- our interest = Menger PM-spaces $(\Omega, \mathcal{F}, \tau_T)$ with triangular function $\tau_T(F, G)(x) = \sup_{u+v=x} T(F(u), G(v))$

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Definition (τ_T -submeasure)

Let $T : [0,1]^2 \to [0,1]$ be a t-norm, and Σ a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ (where $\gamma(E)$ is denoted by γ_E) such that

(a) $\gamma_\emptyset(x) = \varepsilon_0(x)$;

(b) if $E \subset F$, then $\gamma_E(x) \geq \gamma_F(x)$;

(c) $\gamma_{E\cup F}(x+y) \geq T(\gamma_E(x), \gamma_F(y))$, $E, F \in \Sigma$,

is said to be a τ_T -submeasure.

indeed, the notion of τ_T -submeasure is closely related to the Menger PM-space $(\Omega, \mathcal{F}, \tau_T)$ with the 'probabilistic analogue' of the triangle inequality expressed by

$$F_{p,r}(x+y) \geq T(F_{p,q}(x), F_{q,r}(y))$$

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interpretation:

1. τ_T-submeasures = fuzzy number-valued submeasures

2. τ_M-submeasures can be represented by means of a non-decreasing system $(\eta_\alpha)_{\alpha \in [0,1]}$ of numerical submeasures as follows

$$\gamma_E(x) = \sup\{\alpha \in [0,1];\, \eta_\alpha(E) \leq x\}, \quad E \in \Sigma$$

Family of t-norms	**Corresponding family of τ_T-submeasures**
Aczél-Alsina t-norms T_λ^{AA}, $\lambda \in [0, +\infty[$	$\gamma_E^{AA,0}(x) = \varepsilon_{\eta(E)}(x)$ $\gamma_E^{AA,\lambda}(x) = \exp\left(-\left[\max\{\eta(E) - x, 0\}\right]^{1/\lambda}\right)$
Dombi t-norms T_λ^{D}, $\lambda \in [0, +\infty[$	$\gamma_E^{D,0}(x) = \gamma_E^{AA,0}(x)$ $\gamma_E^{D,\lambda}(x) = \left(1 + \left[\max\{\eta(E) - x, 0\}\right]^{1/\lambda}\right)^{-1}$
Frank t-norms T_λ^{F}, $\lambda \in]0, +\infty]$	$\gamma_E^{F,1}(x) = \min\Big\{\exp(x - \eta(E)), 1\Big\}$ $\gamma_E^{F,+\infty}(x) = \max\Big\{\min\{1 + x - \eta(E),\, 1\}, 0\Big\}$ $\gamma_E^{F,\lambda}(x) = \min\left\{\log_\lambda\Big(1 + (\lambda - 1)\exp(x - \eta(E))\Big), 1\right\}$

Family of t-norms	**Corresponding family of τ_T-submeasures**
Hamacher t-norms T^H_λ, $\lambda \in [0,+\infty]$	$\gamma_E^{H,+\infty}(x) = \gamma_E^{AA,0}(x)$ $\gamma_E^{H,0}(x) = \min\left\{\left(1+\eta(E)-x\right)^{-1}, 1\right\}$ $\gamma_E^{H,\lambda}(x) = \min\left\{\lambda\Big(\exp(\eta(E)-x)+\lambda-1\Big)^{-1}, 1\right\}$
Yager t-norms T^Y_λ, $\lambda \in [0,+\infty[$	$\gamma_E^{Y,0}(x) = \gamma_E^{AA,0}(x)$ $\gamma_E^{Y,\lambda}(x) = \max\left\{\min\left\{1-\Big[\max\{\eta(E)-x,0\}\Big]^{1/\lambda}, 1\right\}, 0\right\}$
Sugeno-Weber t-norms T^{SW}_λ, $\lambda \in [-1,+\infty]$	$\gamma_E^{SW,-1}(x) = \gamma_E^{AA,0}(x)$ $\gamma_E^{SW,0}(x) = \gamma_E^{F,+\infty}(x)$ $\gamma_E^{SW,+\infty}(x) = \gamma_E^{F,1}(x)$ $\gamma_E^{SW,\lambda}(x) = \max\left\{\min\left\{\lambda^{-1}\left((1+\lambda)^{1+x-\eta(E)}-1\right), 1\right\}, 0\right\}$

Numerical versus probabilistic submeasure

Numerical $\Rightarrow$ Probabilistic

Let $\eta : \Sigma \to \overline{\mathbb{R}}_+$ be a numerical submeasure. Then

$$\gamma_E(x) = \varepsilon_0(x - \eta(E))$$

is a universal τ_T -submeasure.

interpretation: the number $\gamma_E(x)$ gives the probability that the value of submeasure η of a set $E \in \Sigma$ is less than x

Probabilistic $\Rightarrow$ Numerical

Let T be a t-norm, and Σ be a ring of subsets of Ω. If $\gamma : \Sigma \to \Delta^+$ is a τ_T -submeasure, then the set function $\eta_\gamma : \Sigma \to \overline{\mathbb{R}}_+$ given by

$$\eta_\gamma(E) = \sup\{x \in \overline{\mathbb{R}}_+; \gamma_E(x) < 1\}$$

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Universal probabilistic submeasures

Problem

Characterize a class of all universal probabilistic submeasures!

– to find a function $f : [0, +\infty[\times]0, +\infty[\to [0, 1]$ satisfying the following properties:

(a') $f(0, x) = 1$ for all $x > 0$;

(b') $f(a, x)$ is non-increasing in its first and non-decreasing in its second component;

(c') f is a solution of the functional inequality

$$f(a + b, x + y) \geq \min\{f(a, x), f(b, y)\},$$

for all $a, b \geq 0$, $x, y > 0$.

Constructing universal τ_T-submeasures

Let η be a numerical submeasure on Σ, and $\Phi \in \Delta$. Then a mapping $\gamma : \Sigma \to \Delta^+$ given by

$$\gamma_A(x) = \Phi\left(\frac{cx}{\eta(A)}\right), \quad c > 0, x > 0,$$

is a parametric family of universal τ_T -submeasures.

Examples:

(i) $\gamma_E(x) = 1$ corresponds to $\Phi(z) = \varepsilon_0(z)$

(ii) $\gamma_E(x) = \min\{\frac{cx}{\eta(A)}, 1\}$, $c > 0$, corresponds to a distribution function of random variable uniformly distributed over $[0, 1]$

(iii)

$$\Phi(z) = \begin{cases} \frac{z}{1+z}, & z \geq 0, \\ 0, & \text{otherwise}, \end{cases} \quad \ldots \quad \gamma_E(x) = \frac{cx}{cx + \eta(A)}, \quad c > 0$$

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Examples:

$$\gamma_E(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1/2 & \text{for } x \in]0, \eta(E)]; \\ 1 & \text{for } x > \eta(E), \end{cases}$$

Generated probabilistic submeasures

Constructing τ_T-submeasures from additive generator of T

Let η be a numerical submeasure on Σ. If t is an additive generator of a continuous Archimedean t-norm T, then $\gamma : \Sigma \to \Delta^+$ given by

$$\gamma_E(x) = t^{(-1)}(\eta(E) - x),$$

is a τ_T -submeasure.

Examples: given in table

Probabilistic $\Rightarrow$ Numerical once more

Let γ be a τ_T-submeasure on Σ. If t is an additive generator of a continuous Archimedean t-norm T, then a mapping $\eta_{\gamma,t} : \Sigma \to \mathbb{R}_+$ given by

$$\eta_{\gamma,t}(E) := \sup\{z \in \mathbb{R}_+;\ t(\gamma_E(z)) \geq z\}$$

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Transformations of probabilistic submeasures

Strict t-norms and related probabilistic submeasures

(i) Let φ be an automorphism and γ be a τ_{Π}-submeasure. If φ is supermultiplicative, then γ is a τ_{Π_φ}-submeasure.

(ii) Let T be a strict t-norm with a multiplicative generator θ. Then the following statements are equivalent:

(a) γ is a τ_T -submeasure;

(b) $\theta(\gamma)$ is a τ_{Π}-submeasure.

Nilpotent t-norms and related probabilistic submeasures

(i) Let φ be an automorphism, and γ be a τ_W-submeasure. If $\psi(x) = 1 - \varphi(1 - x)$ is subadditive, then γ is a τ_{W_φ}-submeasure.

(ii) If T is a nilpotent t-norm with additive generator t, then the following statements are equivalent:

(a) γ is a τ_T -submeasure;

(b) $\varphi(\gamma)$ is a τ_W-submeasure, where $\varphi(x) = 1 - \frac{t(x)}{t(0)}$.

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Aggregations of probabilistic submeasures

Aggregation of τ_T-submeasures I

Let T be a continuous Archimedean t-norm with an additive generator t, let $\gamma^{(1)}, \ldots, \gamma^{(n)}$, $n \in \mathbb{N}$, be τ_T -submeasures, and $H : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ be an aggregation function. If there exists a subadditive aggregation function $K : \bigcup_{n\in\mathbb{N}}[0,t(0)]^n \to [0,t(0)]$ such that for all $n \in \mathbb{N}$ and for all $x_i \in [0,1]$, $i = 1,2,\ldots,n$,

$$t(H(x_1,\ldots,x_n)) = K(t(x_1),\ldots,t(x_n)),$$

then $\gamma = H(\gamma^{(1)},\ldots,\gamma^{(n)})$ is a τ_T -submeasure.

Aggregations of probabilistic submeasures

Example: For the strongest subadditive aggregation function $K : \bigcup_{n\in\mathbb{N}}[0, t(0)]^n \to [0, t(0)]$ given by

$$K(u_1, \ldots, u_n) = \begin{cases} 0, & u_1 = \cdots = u_n = 0, \\ t(0), & \text{otherwise}, \end{cases}$$

the condition

$$t(H(x_1, \ldots, x_n)) = K(t(x_1), \ldots, t(x_n))$$

is fulfilled if and only if H is the weakest aggregation function

$$H_w(x_1, \ldots, x_n) = \begin{cases} 1, & x_1 = \cdots = x_n = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\gamma = H_w(\gamma^{(1)}, \ldots, \gamma^{(n)})$ is a τ_T -submeasure.

Aggregations of probabilistic submeasures

Aggregation of τ_T-submeasures II

Let t be an additive generator of a continuous Archimedean t-norm T. If $\gamma^{(i)}$ are τ_T-submeasures for $i = 1, 2, \dots, n$, then $\gamma = \mathbf{A}_t^w\left(\gamma^{(1)}, \dots, \gamma^{(n)}\right)$ is a τ_T -submeasure, where

$$\mathbf{A}_t^w(x_1, \dots, x_n) := t^{(-1)}\left(\sum_{i=1}^{n} w_i\, t(x_i)\right).$$

Aggregation of universal τ_T-submeasures

Let $\gamma^{(1)}, \dots, \gamma^{(n)}$, $n \in \mathbb{N}$, be universal τ_T -submeasures. Since the weakest aggregation function H_w dominates all t-norms, then $\gamma = H_w(\gamma^{(1)}, \dots, \gamma^{(n)})$ is also a universal τ_T -submeasure.

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Problem

Characterize the class of mappings (aggregation functions) which preserve the class of τ_T-submeasures for a fixed T!