IT4Innovations
Centre of Excellence

Relations between Graded Equipollence And Fuzzy C-measures Of Finite Fuzzy Sets

Michal Holčapek

Centre of Excellence IT4Innovations
division of the University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
Ostrava 1, Czech Republic

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OP Výzkum
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Outline

A poor interest about fuzzy cardinal theory

S. Gottwald

Fuzzy uniqueness of fuzzy mappings. *Fuzzy Sets and Systems*, 3:49–74, 1980.

L. Zadeh

A computational approach to fuzzy quantifiers in natural languages. Comp. Math. with Applications **9** (1983) 149–184

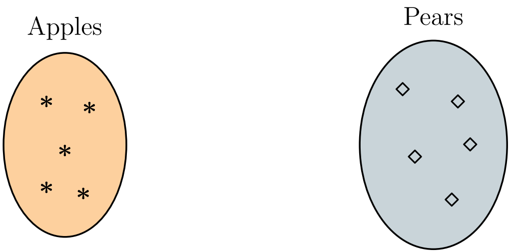
M. Wygralak

Vaguely defined objects. Representations, fuzzy sets and nonclassical cardinality theory. Theory and Decision Library. Kluwer Academic Publisher, 1996.

M. Wygralak

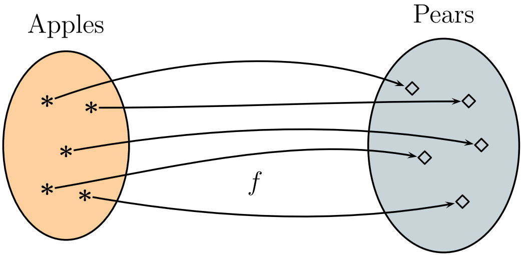
Cardinalities of Fuzzy Sets. Kluwer Academic Publisher, Berlin, 2003.

Example



How to compare the mass of apples and pears?

Functional approach to compare the size of sets

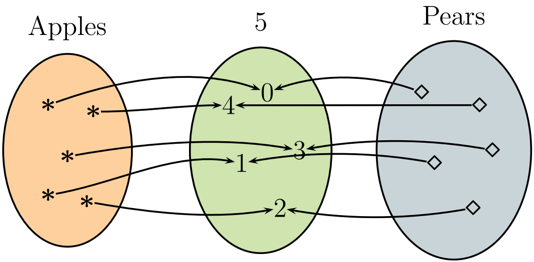


Using a one-to-one correspondence (functional approach).

Approach based on ordinal numbers

Von Neumann construction of natural numbers:

$$0 = \emptyset, 1 = 0 \cup \{0\}, \ldots, 5 = 4 \cup \{4\}, \ldots$$



Using ordinal (cardinal) numbers.

$$|\text{Apples}| = 5 = |\text{Pears}|$$

Two directions in the fuzzy cardinal theory

We can distinguish the approaches based on

1. the relation “to have the same fuzzy cardinality”

$$|A| = |B| \quad \text{or} \quad |A| \sim |B| = \alpha \quad \textit{(graded approach)}$$

2. fuzzy measures similar to the cardinality measure

$$\mathfrak{C}(A) = \textit{real number} \quad \text{or} \quad \mathfrak{C}(A) = \textit{fuzzy number}$$

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i.e.

$$\mathfrak{C} : \mathfrak{Fin} \rightarrow \mathfrak{N}$$

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Several natural questions about $\sim$ and $\mathfrak{C} : \mathfrak{Ffin} \longrightarrow \mathfrak{N}$.

One can ask

- What structure of truth values is suitable? (residuated lattice, MV-algebra, IMTL-algebra???)
- What is $\mathfrak{Ffin}$? (a set or class of fuzzy sets???)
- What is $\mathfrak{N}$? (set or class of finite fuzzy cardinals???)
- How to establish the degree to which two (finite) fuzzy sets have the same cardinality (using one-to-one correspondences between fuzzy sets, or α-cuts???).
- What properties have to keep the mapping $\mathfrak{C}$ to be something like the cardinality measure? (additive measure, cardinality measure for the classical set???)

Residuated-dually residuated lattice

Łukasiewicz algebra

An algebra $([0,1], \wedge, \vee, \otimes, \rightarrow, \oplus, \ominus)$ is the Łukasiewicz algebra, if for $a, b, c \in [0,1]$, we have

- $\neg a = 1 - a$,
- $a \otimes b = \max(a + b - 1, 0)$,
- $a \oplus b = \min(a + b, 1)$ (dual operation to $\otimes$),
- $a \rightarrow b = \min(1 - a + b, 1)$,
- $a \ominus b = \max(a - b, 0)$ (dual operation to $\rightarrow$).

Common denotation

We use $\odot \in \{\wedge, \otimes\}$ and $\overline{\odot} \in \{\vee, \oplus\}$.

Fuzzy sets in the universe of countable sets $\mathfrak{Count}$

Definition

A mapping $A : x \rightarrow L$ is called a countable fuzzy set in $\mathfrak{Count}$, if x is a set in $\mathfrak{Count}$. The class of all countable fuzzy sets in $\mathfrak{Count}$ is denoted by $\mathfrak{Fcount}$.

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Definition

- $\emptyset : \emptyset \to L$ *is the empty fuzzy set,*
- *if* $\mathrm{Dom}(A)$ *contains only one element, then A is a singleton,*
- $\mathrm{Supp}(A) = \{x \in \mathrm{Dom}(A) \mid A(x) > \bot\}$ *is a support of A,*
- *A is a finite fuzzy set, if* $\mathrm{Supp}(A)$ *is a finite set,*
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Equivalence relation for fuzzy sets

Definition

We shall say that fuzzy sets A and B are the equivalent fuzzy sets (symbolically, $A \equiv B$), if $\mathrm{Supp}(A) = \mathrm{Supp}(B)$ and $A(x) = B(x)$ for any $x \in \mathrm{Supp}(A)$.

Definition

$\mathrm{cls}(A)$ *denotes the class of all equivalent fuzzy sets with A.*

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Definition

$\mathrm{cls}(A)$ *denotes the class of all equivalent fuzzy sets with A.*

Operations in $\mathfrak{Fcount}$

Definition

Let $A, B \in \mathfrak{Fcount}$, $x = \mathrm{Dom}(A) \cup \mathrm{Dom}(B)$ *and* $A' \equiv A$, $B' \equiv B$ *such that* $\mathrm{Dom}(A') = \mathrm{Dom}(B') = x$. *Then*

- *the union of* A *and* B *is a mapping* $A \cup B : x \to L$ *defined by*

$$(A \cup B)(a) = A'(a) \vee B'(a),$$

- *the intersection of* A *and* B *is a mapping* $A \cap B : x \to L$ *defined by*

$$(A \cap B)(a) = A'(a) \wedge B'(a),$$

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Example

Consider the Łukasiewicz algebra L

For $A = \{1/a, 0.4/b\}$ and $B = \{0.6/a, 0.2/c\}$ we have

$$A \cup B = \{1/a, 0.4/b, 0.2/c\},$$
$$A \cap B = \{0.6/a, 0/b, 0/c\},$$

Generalized cardinals in FCT for finite fuzzy sets

Definition

A generalized cardinal A (over $\mathbb{N}$) is an $\odot$-convex fuzzy set $A : \mathbb{N} \to L$, i.e.

$$A(i) \odot A(j) \leq A(k), \quad i \leq k \leq j.$$

$\mathfrak{N}$ denotes the set of all generalized cardinals.

Structure of fuzzy cardinals

Addition of fuzzy cardinals and neutral element (zero element)

$$(A+B)(i) = \bigvee_{\substack{k,l\in\mathbb{N}\\ k+l=i}} (A(k)\odot B(l)),$$

$$\mathbf{0}(k) = \begin{cases} 1, & \text{k=0;} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

The triplet $(\mathfrak{N}, +, \mathbf{0})$ *is a commutative monoid.*

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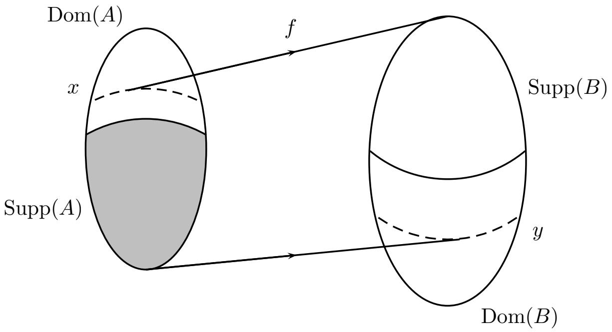
How to define degrees of one-to-one mappings

Definition

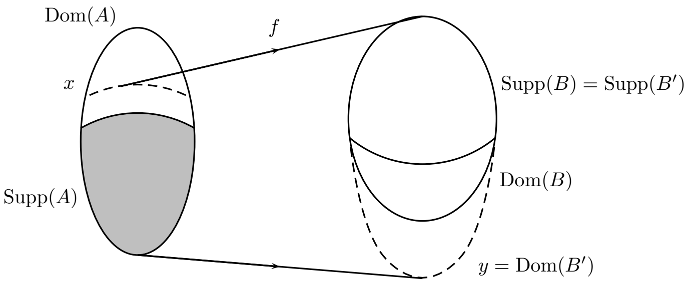
Let $A, B \in \mathfrak{Ffin}$, $x, y \in \mathfrak{Count}$ and $f : x \to y$ be a one-to-one mapping of x onto y in $\mathfrak{Count}$. We shall say that f is a one-to-one mapping of A onto B in the degree α with respect to $\odot$, if $\mathrm{Supp}(A) \subseteq x \subseteq \mathrm{Dom}(A)$ and $\mathrm{Supp}(B) \subseteq y \subseteq \mathrm{Dom}(B)$ and

$$\alpha = \bigodot_{z \in x} (A(z) \leftrightarrow B(f(z))).$$

How does it work?



But we can imagine much more!



How to define a graded equipollence of countable fuzzy sets

Definition

Let $A, B \in \mathfrak{Fcount}$. *A mapping* $f : x \to y$ *belongs to the set* $\mathrm{Bij}(A, B)$, *if*

(i) f *is a one-to-one mapping of* x *onto* y,

(ii) $\mathrm{Supp}(A) \subseteq x \subseteq \mathrm{Dom}(A)$, *and*

(iii) $\mathrm{Supp}(B) \subseteq y \subseteq \mathrm{Dom}(B)$.

Definition of graded equipollence between countable fuzzy sets

Definition

Let $A, B \in \mathfrak{Fcount}$. We shall say that A is equipollent with B (or A has the same cardinality as B) in the degree α, if there exist fuzzy sets $C \in \mathrm{cls}(A)$ and $D \in \mathrm{cls}(B)$ such that

$$\alpha = \bigvee_{f \in \mathrm{Bij}(C,D)} [C \sim_f^{\odot} D]$$

and, for each $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$ and $f \in \mathrm{Bij}(A', B')$, there is $[A' \sim_f B'] \leq \alpha$.

Graded equipollence for finite fuzzy sets

Theorem

Let $A, B \in \mathfrak{Ffin}$ *and* $C \in \mathrm{cls}(A)$, $D \in \mathrm{cls}(B)$ *be such that*

$$z = \mathrm{Dom}(C) = \mathrm{Dom}(D) \quad \text{and} \quad |z| = m.$$

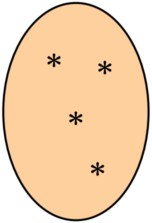
Then

$$[A \sim^{\odot} B] = \bigvee_{f \in \mathrm{Perm}(z)} [C \sim_f^{\odot} D],$$

where $\mathrm{Perm}(z)$ *denotes the set of all permutations on* z*.*

How to model the behavior of fuzzy "cardinality" measures?

$A =$ "Apples"



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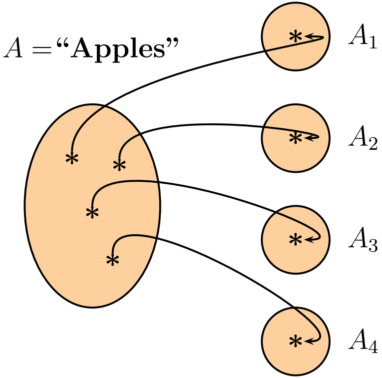

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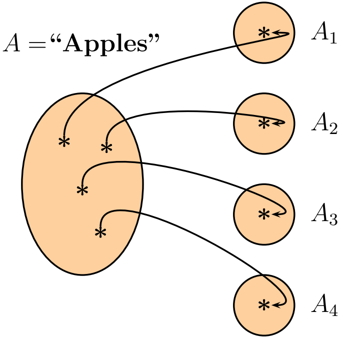

$$|A| = |A_1| + |A_2| + |A_3| + |A_4| = 4$$

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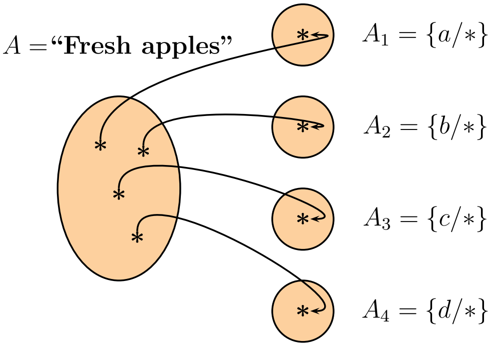

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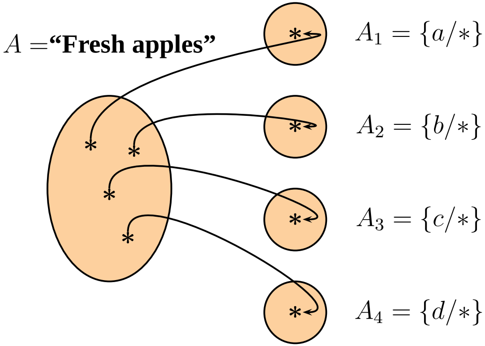

$$\boxed{\mathfrak{C}(A_i)(n) = ?,\ n \in \mathbb{N}}$$

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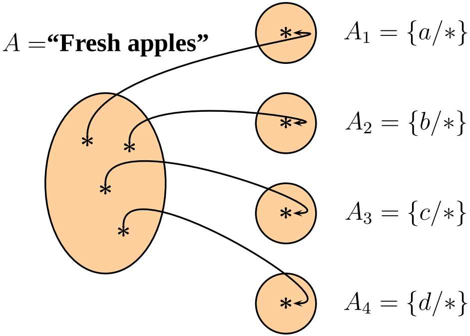

$$\mathfrak{C}(A_i)(n) = ?,\ n \in \mathbb{N}$$

$$\mathfrak{C}(A) = \mathfrak{C}(A_1) + \mathfrak{C}(A_2) + \mathfrak{C}(A_3) + \mathfrak{C}(A_4)$$

$\mathfrak{C}(A_i)(n) = ?$

Crisp set	Fuzzy set
$\mathfrak{C}(\{1/*\})(0) = 0$	$\mathfrak{C}(\{a/*\})(0) = \alpha$
$\mathfrak{C}(\{1/*\})(1) = 1$	$\mathfrak{C}(\{a/*\})(1) = \beta$
$\mathfrak{C}(\{1/*\})(2) = 0$	$\mathfrak{C}(\{a/*\})(2) = 0$
$\mathfrak{C}(\{1/*\})(3) = 0$	$\mathfrak{C}(\{a/*\})(3) = 0$
$\vdots$	$\vdots$

Explanation of α and β.

$\alpha =$ "the degree of non-existence of $*$ in $A_1 = \{a/*\}$".
$\beta =$ "the degree of existence of $*$ in $A_1 = \{a/*\}$".

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Definition

A class mapping $\mathfrak{C} : \mathfrak{Ffin} \to \mathfrak{N}$ *is a* *fuzzy c-measure of finite fuzzy sets* *with respect to* $\odot$*, if, for arbitrary* $A, B \in \mathfrak{Ffin}$*, it holds:*

C1: *if* $\mathrm{Supp}(A) \cap \mathrm{Supp}(B) = \emptyset$*, then* $\mathfrak{C}(A \cup B) = \mathfrak{C}(A) + \mathfrak{C}(B)$*,*

C2: *if* $i, j \in \mathbb{N}$ *and* $i > |\mathrm{Supp}(A)|$*,* $j > |\mathrm{Supp}(B)|$*, then* $\mathfrak{C}(A)(i) = \mathfrak{C}(B)(j)$*,*

C3: *if* A *is a crisp set, then* $\mathfrak{C}(A)$ *is a crisp set,* $\mathfrak{C}(A)(|A|) = \top$*,*

C4: *if* $a \in L$ *and* $x, y \in \mathfrak{Count}$*, then* $\mathfrak{C}(\{a/x\}) = \mathfrak{C}(\{a/y\})$*,*

C5: *if* $a, b \in L$ *and* $x \in \mathfrak{Count}$*, then*

$$\mathfrak{C}(\{a \overline{\odot} b/x\})(0) = \mathfrak{C}(\{a/x\})(0) \odot \mathfrak{C}(\{b/x\})(0),$$
$$\mathfrak{C}(\{a \odot b/x\})(1) = \mathfrak{C}(\{a/x\})(1) \odot \mathfrak{C}(\{b/x\})(1).$$

Example

Consider

$$\mathfrak{C}_{id}(A)(i) = \mathrm{FGCount}(A)(i) = \bigvee\{a \mid a \in L \text{ and } |A_a| \geq i\}$$

and define

$$\mathfrak{C}(A)(i) = \begin{cases} \top, & i = 0, \\ \mathfrak{C}(A)(i-1) \otimes \mathfrak{C}_{id}(A)(i), & \text{otherwise.} \end{cases}$$

For $A = \{0.5/a, 0.8/b, 0.1/c, 0.4/d, 0/e\}$, we obtain

$$\mathfrak{C}(A) = \{1/0, 0.8/1, 0.3/2, 0/3, 0/4, 0/5, 0/6, \ldots\},$$

where e.g. $\mathfrak{C}(A)(2) = 0.8 \otimes 0.5 = \max(0.8 + 0.5 - 1, 0) = 0.3$.

Theorem (Representation of c-measures)

Let $\mathfrak{C} : \mathfrak{Fin} \to \mathfrak{N}$ be a mapping satisfying the additivity axiom and $\mathfrak{C}(A) = \mathfrak{C}(\emptyset)$ for any $A \in \mathrm{cls}(\emptyset)$. Then the following statements are equivalent:

(i) *$\mathfrak{C}$ is a c-measure of finite fuzzy sets with respect to $\odot$,*

(ii) *there exist an $\odot$-homomorphism $f : L \to L$ and an $\overline{\odot}_{\mathrm{d}}$-homomorphism $g : L \to L$, such that $f(\bot) \in \{\bot, \top\}$, $g(\top) \in \{\bot, \top\}$ and*

$$\mathfrak{C}(\{a/x\})(0) = g(a), \ \mathfrak{C}(\{a/x\})(1) = f(a),$$
$$\mathfrak{C}(\{a/x\})(k) = f(\bot), \ k > 1$$

hold for arbitrary $a \in L$ and $x \in \mathfrak{Count}$.

Denote $\mathfrak{C}_{g,f}$ a c-measure determined by g and f.

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Denote $\mathfrak{C}_{g,f}$ a c-measure determined by g and f.

Corollary

Let **L** *be a linearly ordered rdr-lattice,* $\mathfrak{C}_{g,f}$ *be a c-measure such that* f *is a* $\odot$*-po-homomorphism and* g *is a* $\overline{\odot}_{\mathrm{d}}$*-po-homomorphism. Then*

$$\mathfrak{C}_{g,f}(A)(i) = \mathfrak{C}_g(A)(i) \odot \mathfrak{C}_f(A)(i)$$

holds for any $A \in \mathfrak{F}\mathfrak{fin}$ *and* $i \in \mathbb{N}$.

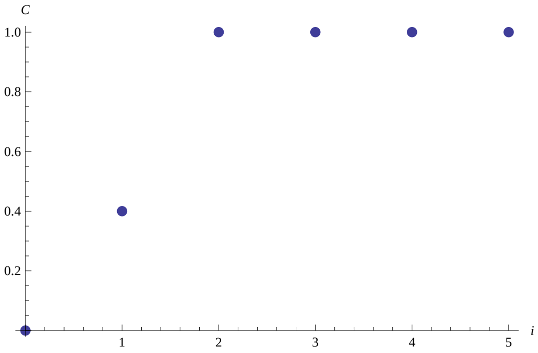
Corollary

Let $\mathbf{L}$ *be linearly ordered,* $\mathfrak{C}_{g,f}$ *be a c-measure with respect to* $\wedge$ *such that* f *is* $\wedge$*-homomorphism and* g *is* $\vee$*-homomorphisms. Then*

$$\mathfrak{C}_{g,f}(A)(i) = g(\mathfrak{C}_{id}(A)(i+1)) \wedge f(\mathfrak{C}_{id}(A)(i))$$

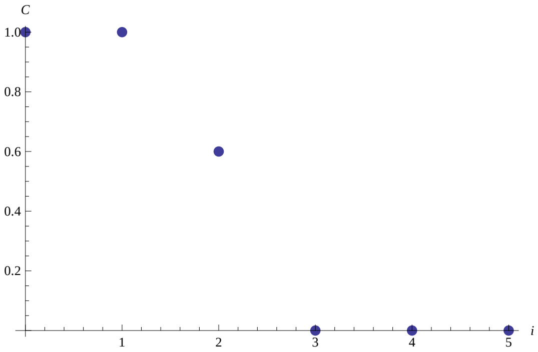
holds for any $A \in \mathfrak{F}\mathfrak{fin}$ *and* $i \in \mathbb{N}$.

$\mathfrak{C}_g(A)$ for $A = \{0.6/x, 1/y\}$ and $g(x) = 1 - x$



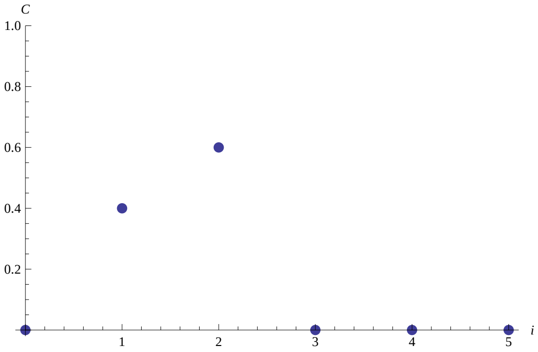
$\mathfrak{C}_g(A)(i) =$ "at most i elements in A"

$\mathfrak{C}_f(A)$ for $A = \{0.6/x, 1/y\}$ and $f(x) = x$



$\mathfrak{C}_g(A)(i) =$ "at least i elements in A"

$\mathfrak{C}_{g,f}(A)$ for $A = \{0.6/x, 1/y\}$ and $f(x) = x$, $g(x) = 1 - x$



$\mathfrak{C}_{g,f}(A)(i) =$ "exactly i elements in A"

Denote

$f(A) = f \circ A$

Definition

We shall say that fuzzy sets A and B are the equivalent fuzzy sets in the degree a (symbolically, $[A \approx B] = a$*), if*

$$a = \bigwedge_{x \in \mathrm{Dom}(A) \cup \mathrm{Dom}(B)} (A'(x) \leftrightarrow B'(x)),$$

holds for $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$ *with*

$$\mathrm{Dom}(A') = \mathrm{Dom}(B') = \mathrm{Dom}(A) \cup \mathrm{Dom}(B).$$

Theorem

Let $\mathfrak{C}_{g,f}$ *be a c-measure. Then*

$$[g(A) \sim_h^{\odot} g(B)] \odot [f(A) \sim_h^{\odot} f(B)] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)]$$

holds for any $A, B \in \mathfrak{Fin}$ *such that* $|\mathrm{Dom}(A)| = |\mathrm{Dom}(B)| = m$ *and* $h \in \mathrm{Perm}(A, B)$*.*

Corollary

Let $\mathfrak{C}_{g,f}$ be a c-measure. Then

(i) $[g(A) \sim^{\odot} g(B)] \leq [\mathfrak{C}_g(A) \approx \mathfrak{C}_g(B)]$

(ii) $[f(A) \sim^{\odot} f(B)] \leq [\mathfrak{C}_f(A) \approx \mathfrak{C}_f(B)]$,

hold for any $A, B \in \mathfrak{Fin}$ such that $|\mathrm{Dom}(A)| = |\mathrm{Dom}(B)| = m$.

Corollary

Let $\mathfrak{C}_{g,f}$ *be a c-measure. Then*

(i) $[g(A) \sim^{\odot} g(B)] \leq [\mathfrak{C}_g(A) \approx \mathfrak{C}_g(B)]$

(ii) $[f(A) \sim^{\odot} f(B)] \leq [\mathfrak{C}_f(A) \approx \mathfrak{C}_f(B)]$,

hold for any $A, B \in \mathfrak{Fin}$ *such that* $|\mathrm{Dom}(A)| = |\mathrm{Dom}(B)| = m$.

Theorem

Let **L** *be a linearly ordered rdr-lattice,* $\mathfrak{C}_{g,f}$ *be a c-measure such that* f *is a* $\odot$*-po-homomorphism and* g *is a* $\overline{\odot}_{\mathrm{d}}$*-po-homomorphism. Then*

$$[g(A) \sim^{\odot} g(B)] \odot [f(A) \sim^{\odot} f(B)] \leq [\mathfrak{C}_{g,f}(A) \approx \mathfrak{C}_{g,f}(B)]$$

for any $A, B \in \mathfrak{Fin}$. *Especially, if* $\mathfrak{C}_g$ *and* $\mathfrak{C}_f$ *are c-measures with respect to* $\odot = \wedge$, *then*

(i) $[g(A) \sim^{\wedge} g(B)] = [\mathfrak{C}_g(A) \approx \mathfrak{C}_g(B)]$,

(ii) $[f(A) \sim^{\wedge} f(B)] = [\mathfrak{C}_f(A) \approx \mathfrak{C}_f(B)]$

hold for any $A, B \in \mathfrak{Fin}$ *such that* $|\mathrm{Dom}(A)| = |\mathrm{Dom}(B)| = m$.

A future work

- To investigate further properties of fuzzy c-measures of finite fuzzy sets.
- To investigate further relations between fuzzy c-measures and graded equipollence of finite fuzzy sets.
- To extend c-measures to infinite case.
- To develop the fuzzy cardinality theory.

⋮

Thank you for your attention.