# The theory of functions on IF-sets

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- Introduction (basic notions)
- Relations cyclometric and goniometric functions applied on IF set
- Diferencial calculus
  - The derivative of inverse function
  - The derivative of a composite function
  - Higher-order derivatives

# Introduction (basic notions)

## IF-events

- $(\Omega, S)$ is a measurable space
- $\mu_A, \nu_A$ are S-measurable functions
- $\mathcal{F}$=$\{A = (\mu_A, \nu_A), \mu_A, \nu_A : \Omega \to \langle 0, 1 \rangle, \mu_A + \nu_A \leq 1\}$
  is family of all IF-events

# Introduction (basic notions)

**$\ell$-group $\mathcal{G}$**

- $\mathcal{G} = \{A = (\mu_A, \nu_A) ; \mu_A : \Omega \to \mathbf{R}, \nu_A : \Omega \to \mathbf{R}\}$
- For any $A, B \in \mathcal{G}$
  - $A + B = (\mu_A, \nu_A) + (\mu_B, \nu_B) = (\mu_A + \mu_B, 1 - (1 - \nu_A + 1 - \nu_B)) = (\mu_A + \mu_B, \nu_A + \nu_B - 1)$
  - $A \leq B \Leftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B$

$\mathcal{G} = (\mathcal{G}, +, \leq)$

## Introduction (basic notions)

### Basic algebraic operations on $\mathcal{G}$

- $A - B = (\mu_A, \nu_A) - (\mu_B, \nu_B) = (\mu_A - \mu_B, 1 - (1 - \nu_A - 1 + \nu_B)) = (\mu_A - \mu_B, \nu_A - \nu_B + 1)$
- $A \cdot B = (\mu_A, \nu_A)(\mu_B, \nu_B) = (\mu_A \cdot \mu_B, 1 - ((1 - \nu_A)(1 - \nu_B))) = (\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B)$
- $\frac{A}{B} = \frac{(\mu_A, \nu_A)}{(\mu_B, \nu_B)} = \left(\frac{\mu_A}{\mu_B}, 1 - \frac{1 - \nu_A}{1 - \nu_B}\right) = \left(\frac{\mu_A}{\mu_B}, \frac{\nu_A - \nu_B}{1 - \nu_B}\right), B \neq (0, 1)$
- neutral element $\mathbf{0} = (0, 1)$
- inverse element of $A$, i.e. $-A = (-\mu_A, 2 - \nu_A)$
- united element $\mathbf{1} = (1, 0)$

# Functions on $\mathcal{G}$

**Definition (Riečan, B. and Hollá, I.: *Elementary functions on IF sets*)**

Considering a function $f : \mathbf{R} \to \mathbf{R}$. Then $f(A) \in \mathcal{G}$ and

$$f(A) = f(\mu_A, \nu_A) = (f(\mu_A), 1 - f(1 - \nu_A))$$

for any $A \in \mathcal{G}$.

# Goniometric functions

**Definition** (Riečan, B. and Hollá, I.: *Elementary functions on IF sets*)

For any $A \in \mathcal{G}$ we define:

$sin\ A = (sin\,\mu_A, 1 - sin\,(1 - \nu_A))$

$cos\ A = (cos\,\mu_A, 1 - cos\,(1 - \nu_A))$

$tg\ A = (tg\,\mu_A, 1 - tg\,(1 - \nu_A))\,, \mu_A \neq \frac{\pi}{2} + 2k\pi, 1 - \nu_A \neq \frac{\pi}{2} + 2k\pi$

$cotg\ A = (cotg\,\mu_A, 1 - cotg\,(1 - \nu_A))\,, \mu_A \neq 2k\pi, 1 - \nu_A \neq 2k\pi$

# Goniometric functions

**Definition**

For any $A \in \mathcal{G}$ we define:

$sin\ A = (sin\ \mu_A, 1 - sin\,(1 - \nu_A))\,, \left(-\frac{\pi}{2}, 1 + \frac{\pi}{2}\right) \leq A \leq \left(\frac{\pi}{2}, 1 - \frac{\pi}{2}\right)$

$cos\ A = (cos\ \mu_A, 1 - cos\,(1 - \nu_A))\,, (0, 1) \leq A \leq (\pi, 1 - \pi)$

$tg\ A = (tg\ \mu_A, 1 - tg\,(1 - \nu_A))\,, \left(-\frac{\pi}{2}, 1 + \frac{\pi}{2}\right) < A < \left(\frac{\pi}{2}, 1 - \frac{\pi}{2}\right)$

$cotg\ A = (cotg\ \mu_A, 1 - cotg\,(1 - \nu_A))\,, (0, 1) < A < (\pi, 1 - \pi)$

# Cyclometric functions

**Definition**

For any $A \in \mathcal{G}$ we define:

$arcsin\ A = (arcsin\ \mu_A, 1 - arcsin\,(1 - \nu_A))\,, (-1, 2) \leq A \leq (1, 0)$

$arccos\ A = (arccos\ \mu_A, 1 - arccos\,(1 - \nu_A))\,, (-1, 2) \leq A \leq (1, 0)$

$arctg\ A = (arctg\ \mu_A, 1 - arctg\,(1 - \nu_A))\,, (-1, 2) \leq A \leq (1, 0)$

$arccotg\ A = (arccotg\ \mu_A, 1 - arccotg\,(1 - \nu_A))\,, (-1, 2) \leq A \leq (1, 0)$

# Cyclometric functions

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$arccotg\ A = (arccotg\ \mu_A, 1 - arccotg\,(1 - \nu_A))\,, (-1, 2) \leq A \leq (1, 0)$

**Theorem**

*arcsin* $A \in \mathcal{F}$ *for any* $A \in \mathcal{F}$

**Theorem**

*arctg* $A \in \mathcal{F}$ *for any* $A \in \mathcal{F}$

# Relations cyclometric and goniometric functions

## Theorem

$\forall A \in \mathcal{G}$, *such that* $A = (\mu_A, \nu_A)$ :

a) $arcsin\,(sin\,A) = A, \left(-\frac{\pi}{2}, 1 + \frac{\pi}{2}\right) \leq A \leq \left(\frac{\pi}{2}, 1 - \frac{\pi}{2}\right)$

b) $sin\,(arcsin\,A) = A, (-1, 2) \leq A \leq (1, 0)$

c) $arccos\,(cos\,A) = A, (0, 1) \leq A \leq (\pi, 1 - \pi)$

d) $cos\,(arccos\,A) = A, (-1, 2) \leq A \leq (1, 0)$

e) $arctg\,(tg\,A) = A, \left(-\frac{\pi}{2}, 1 + \frac{\pi}{2}\right) < A < \left(\frac{\pi}{2}, 1 - \frac{\pi}{2}\right)$

f) $tg\,(arctg\,A) = A, (-1, 2) \leq A \leq (1, 0)$

g) $arccotg\,(cotg\,A) = A, (0, 1) < A < (\pi, 1 - \pi)$

h) $cotg\,(arccotg\,A) = A, (-1, 2) \leq A \leq (1, 0)$

# Relations cyclometric and goniometric functions

**Theorem**

$\forall A \in \langle(-1,2),(1,0)\rangle$, *such that* $A = (\mu_A, \nu_A)$ :

a) $arcsin\,(-A) = -arcsin\; A$

b) $arctg\,(-A) = -arctg\; A$

# Relations cyclometric and goniometric functions

**Theorem**

$\forall A, B \in \langle (-1, 2), (0, 1) \rangle$ *,such that* $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ :

a) $arcsin\ A + arcsin\ B = arcsin\left(A\sqrt{1 - B^2} + B\sqrt{1 - A^2}\right)$

b) $arcsin\ A - arcsin\ B = arcsin\left(A\sqrt{1 - B^2} - B\sqrt{1 - A^2}\right)$

c) $arccos\ A + arccos\ B = arccos\left(AB - \sqrt{1 - A^2}\sqrt{1 - B^2}\right)$

d) $arccos\ A - arccos\ B = -arccos\left(AB + \sqrt{1 - A^2}\sqrt{1 - B^2}\right)$

$arccos\ A - arccos\ B = arccos\left(AB + \sqrt{1 - A^2}\sqrt{1 - B^2}\right), ifA < B$

# Relations cyclometric and goniometric functions

**Theorem**

e) $arctg\ A + arctg\ B = arctg \dfrac{A+B}{\mathbf{1} - AB}$

f) $arctg\ A - arctg\ B = arctg \dfrac{A-B}{\mathbf{1} + AB}$

g) $arccotg\ A + arccotg\ B = arccotg \dfrac{AB - \mathbf{1}}{A+B}, A \neq -B$

h) $arccotg\ A - arccotg\ B = arccotg \dfrac{AB + \mathbf{1}}{B-A}, A \neq B$

# The differential calculus

# The differential calculus

**Definition** (Michalíková, A.: *The differential calculus on IF sets*. 2009)

Denote $\tilde{\epsilon} = (\epsilon, 1-\epsilon)$ and $\tilde{\delta} = (\delta, 1-\delta)$. Let $f$ be a function defined on the $\ell$-group $\mathcal{G}$ and let $A_0, A, L, \tilde{\epsilon}, \tilde{\delta}$ be from $\mathcal{G}$. For a function $f$ of a variable $A$ defined on a neighborhood of a point $A_0$. If

$$\forall \tilde{\epsilon} > (0,1)\, \exists \tilde{\delta} > (0,1)\, \forall A_0 \in \left(A - \tilde{\delta}, A + \tilde{\delta}\right) \setminus \{A\}\, ; f\,(A_0) \in (L - \tilde{\epsilon}, L + \tilde{\epsilon})\,,$$

then we say that $L$ is the limit of function $f$ at the point $A$ and write

$$\lim_{A \to A_0} f\,(A) = L.$$

# The differential calculus

**Definition** (Michalíková, A.: *The differential calculus on IF sets.* 2009)

Let function $f$ be defined on a neighborhood of a point $A_0$ and let $A_0, A, L, \tilde{\epsilon}, \tilde{\delta}$ be from $\mathcal{G}$. Let

$$\lim_{A \to A_0} \frac{f(A) - f(A_0)}{A - A_0}$$

exist. Then this limit is the derivative of the function $f$ at the point $A_0$ and we will denote it as $f'(A_0)$.

**Definition (Michalíková, A.: *The differential calculus on IF sets*. 2009)**

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$$\lim_{A \to A_0} \frac{f(A) - f(A_0)}{A - A_0}$$

exist. Then this limit is the derivative of the function $f$ at the point $A_0$ and we will denote it as $f'(A_0)$.

**Theorem**

$f'(A_0) = (f'(\mu_{A_0}), 1 - f'(1 - \nu_{A_0}))$

# The differential calculus

**Definition** (Michalíková, A.: *Elementary functions on IF sets.* 2009)

The function $f$ is continuous at the point $A_0$ if and only if

$$\lim_{A \to A_0} f(\mu_A) = f(\mu_{A_0}),$$

and at the same time

$$\lim_{A \to A_0} f(1 - \nu_A) = f(1 - \nu_{A_0}).$$

**Definition** (Michalíková, A.: *The differential calculus on IF sets.* 2009)

Let the function $f$ has the derivative at the point $A_0$. Then the function $f$ is continuous at the point $A_0$.

# The differential calculus

## The derivative of inverse function

Let functions $f, \varphi$ are continuous, monotonous and inverse to each other $\left(\varphi = f^{-1}\right)$ and let $A_0, B_0 \in \mathcal{G}$ such that

$$f\left(A_0\right) = B_0 \Leftrightarrow \varphi\left(B_0\right) = A_0,$$

for all $A_0 \in \mathcal{D}\left(f\right)$.
Let exist derivative of function $\varphi$, such that $\varphi^{-1}\left(B_0\right) \neq (0, 1)$.Then exist derivative of function $f$ at the point $B_0$:

$$\begin{aligned} f'\left(A_0\right) &= \frac{\mathbf{1}}{\varphi'\left(B_0\right)} = \left(\frac{1}{\varphi'\left(\mu_{B_0}\right)}, 1 - \frac{1}{\varphi'\left(1-\nu_{B_0}\right)}\right) \\ &= \left(\frac{1}{\left[f^{-1}\left(\mu_{B_0}\right)\right]'}, 1 - \frac{1}{\left[f^{-1}\left(1-\nu_{B_0}\right)\right]'}\right). \end{aligned}$$

# The differential calculus

**Theorem**

a) $f(A) = \ln A \Rightarrow \forall A_0 > \boldsymbol{0} \quad f'(A_0) = \frac{\boldsymbol{1}}{A_0}$

b) $f(A) = arcsin\, A \Rightarrow \forall A_0 \in \langle(-1,2),(0,1)\rangle \quad f'(A_0) = \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{1} - A_0}}$

c) $f(A) = arccos\, A \Rightarrow \forall A_0 \in \langle(-1,2),(0,1)\rangle \quad f'(A_0) = -\frac{\boldsymbol{1}}{\sqrt{\boldsymbol{1} - A_0}}$

d) $f(A) = arctg\, A \Rightarrow \forall A_0 \in \langle(-1,2),(0,1)\rangle \quad f'(A_0) = \frac{\boldsymbol{1}}{A_0^2 + \boldsymbol{1}}$

e) $f(A) = arccotg\, A \Rightarrow \forall A_0 \in \langle(-1,2),(0,1)\rangle \quad f'(A_0) = -\frac{\boldsymbol{1}}{A_0^2 + \boldsymbol{1}}$

# The differential calculus

**Theorem: The derivative of a composite function**

Let functions $f, g : \mathbf{R} \to \mathbf{R}$ and a point $A_0 \in \mathcal{G}$. Let $f'(g(A_0))$ exist. This derivative is the derivative of function $f$ at the point $g(A_0)$. Then exist derivative of function $h(A) = f(g(A))$ and

$$h'(A_0) = f'(g(A_0)) \cdot g'(A_0).$$

# The differential calculus - higher order derivatives

**Definition**

Let function $f$ be defined on a neighborhood of a point $A_0$ and let $A_0 \in \mathcal{G}$. Let the function $f$ has the derivative at the point $A_0$. Let

$$\lim_{A \to A_0} \frac{f'(A) - f'(A_0)}{A - A_0}$$

exist. Then this limit is the second derivative of the function $f$ at the point $A_0$ and we will denote it as $f''(A_0)$.

# The differential calculus - higher order derivatives

## Definition

Let function $f : \mathbf{R} \to \mathbf{R}$. We denote $f^{(0)}(A_0) = f(A_0)$. Then

$$f^{(n)}(A_0) = \lim_{A \to A_0} \frac{f^{(n-1)}(A) - f^{(n-1)}(A_0)}{A - A_0} = \left[ f^{(n-1)}(A_0) \right]'.$$

This number is the *n-th derivative* of the function $f$ at the point $A_0$.

## Theorem

$$f^{(n)}(A_0) = \left( f^{(n)}(\mu_{A_0}), 1 - f^{(n)}(1 - \nu_{A_0}) \right)$$

Thank you.