On probabilistic submeasures

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Introduction and motivations

- probabilistic submeasures as non-additive set functions have appeared naturally in classical measure theory
- much attention was paid there to develop a theory of submeasures
  $\longrightarrow$ for example, Dobrakov submeasures and semimeasures and their various generalizations and extensions
- we have only a probabilistic information about measure of a set
  $\longrightarrow$ for example, if rounding of reals is considered, then the uniform distributions over intervals describe our information about the measure of a set
- interpretation as a fuzzy number
  $\longrightarrow$ the value γ_E can be seen as a non-negative LT-fuzzy number, where $\tau_T(\gamma_E, \gamma_F)$ corresponds to the T-sum of fuzzy numbers γ_E and γ_F

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Basic notations

- ***distance distribution function*** $F : \overline{\mathbb{R}} \to [0,1]$ is non-decreasing, left continuous on $\mathbb{R}$, $F(-\infty) = 0$, $F(+\infty) = 1$ and $F(0) = 0$

  the class of all distance distribution functions will be denoted by Δ^+

  for example, the distribution function of Dirac random variable concentrated in point 0

$$\varepsilon_0(x) = \begin{cases} 0, & \text{for } x \leq 0, \\ 1, & \text{for } x > 0 \end{cases}$$

- ***triangle function*** is a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ which is symmetric, associative, non-decreasing in each variable and has ε_0 as the identity

- ***triangular norm*** is a mapping $T : [0,1]^2 \to [0,1]$ which is symmetric, associative, non-decreasing in each argument and has 1 as the identity

Definition (τ_T -submeasure)

Let $T : [0,1]^2 \to [0,1]$ be a t-norm, and Σ a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ (where $\gamma(E)$ is denoted by γ_E) such that

(a) if $E = \emptyset$, then $\gamma_\emptyset(x) = \varepsilon_0(x)$, $x > 0$;

(b) if $E \subset F$, then $\gamma_E(x) \geq \gamma_F(x)$, $x > 0$;

(c) $\gamma_{E \cup F}(x + y) \geq T(\gamma_E(x), \gamma_F(y))$, $\ x, y > 0, E, F \in \Sigma$,

is said to be a τ_T *-submeasure*.

- the notion of τ_T -submeasure is closely related to the Menger PM-space $(\Omega, \mathcal{F}, \tau_T)$ where τ_T is the triangle function in the form

$$\tau_T(G,H)(x) = \sup_{u+v=x} T(G(u), H(v)),$$

and T is a left-continuous t-norm

Definition ($\tau_{L,T}$ -submeasure)

In effort to generalize the concept of τ_T -submeasure let us introduce the following notations

- Let us denote by $\mathcal{L}$ the set of binary operations on $\overline{\mathbb{R}}_+$ such that
  - (a) L is commutative and associative;
  - (b) L is jointly strictly increasing, i.e., for all $u_1, u_2, v_1, v_2 \in \overline{\mathbb{R}}_+$ with $u_1 < u_2$, $v_1 < v_2$ holds $L(u_1, v_1) < L(u_2, v_2)$;
  - (c) L is continuous on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$;
  - (d) L has 0 as its neutral element.
- Let T is a left-continuous t-norm and $L \in \mathcal{L}$. Then for all $x \in \overline{\mathbb{R}}_+$ and $G, H \in \Delta^+$

$$\tau_{L,T}(G,H)(x) = \sup_{L(u,v)=x} T(G(u), H(v))$$

is a triangle function.

Definition ($\tau_{L,T}$ -submeasure)

Let $(L, T) \in \mathcal{L} \times \mathcal{T}$ and Σ be a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ such that

(a') $\gamma_\emptyset(x) = \varepsilon_0(x)$, $x > 0$;

(b') $\gamma_E(x) \geq \gamma_F(x)$, $x > 0$ whenever $E \subset F$;

(c') $\gamma_{E\cup F}(L(x,y)) \geq T(\gamma_E(x), \gamma_F(y))$, $x, y > 0$, $E, F \in \Sigma$,

is said to be a $\tau_{L,T}$ *-submeasure*.

- if L is classical addition, we simply speak about τ_T -submeasure
- the order $\ll$ on the set of all $\tau_{L,T}$-submeasures $\Theta_{\mathcal{L},\mathcal{T}}$
- pseudo-metrics and metrics generated by $\tau_{T,L}$-submeasures

A few examples of $\tau_{L,T}$-submeasures

Let $\eta : \Sigma \to \overline{\mathbb{R}}_+$ be a numerical submeasure on a ring Σ of a non-empty set Ω and $E \in \Sigma$. Then

(a) $\gamma \in \Theta_{L,M}$, where $L \in \mathcal{L}$, $L \geq K_1$ and

$$\gamma_E(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1/2 & \text{for } x \in]0, \eta(E)]; \\ 1 & \text{for } x > \eta(E), \end{cases}$$

(b) $\gamma \in \Theta_{L,D}$, where $L \in \mathcal{L}$ is arbitrary and

$$\gamma_E(x) = \frac{x}{x + \eta(E)};$$

(c) $\gamma \in \Theta_{L,W}$, where $L \in \mathcal{L}$, $L \geq K_1$ and

$$\gamma_E(x) = \max\Big\{\min\{1 + x - \eta(E), 1\}, 0\Big\}.$$

Poset of all $\tau_{L,T}$-submeasures

We use the usual point-wise order $\leq$ between real-valued functions:

- since $\gamma \in \Delta^+$ is non-decreasing, then for a fixed $T \in \mathcal{T}$ each $\tau_{L_1,T}$-submeasure is a $\tau_{L_2,T}$-submeasure whenever $L_1 \leq L_2$.
- moreover, if $T_2 \leq T_1$ then each τ_{L_1,T_1}-submeasure is a τ_{L_2,T_2}-submeasure.

$\Downarrow$

We define the order $\ll$ on $\Theta_{\mathcal{L},\mathcal{T}}$ as follows

$$\Theta_{L_1,T_1} \ll \Theta_{L_2,T_2} \quad \text{if and only if} \quad L_1 \leq L_2 \text{ and } T_2 \leq T_1.$$

Then $(\Theta_{\mathcal{L},\mathcal{T}}, \ll)$ is a partially ordered set (poset) and since for any t-norm T we have $M \geq T \geq D$, then for each $(L, T) \in \mathcal{L} \times \mathcal{T}$ holds

$$\Theta_{L,M} \ll \Theta_{L,T} \ll \Theta_{L,D}.$$

Pseudo-metrics and metrics generated by $\tau_{L,T}$-submeasures

Proposition

Let $L \in \mathcal{L}$ be an operation on $\overline{\mathbb{R}}_+$ such that $L \leq K_1$. If γ is a $\tau_{L,T}$-submeasure on Σ, then the function

$$\chi(E,F) = \sup\{x > 0;\ \gamma_{E\triangle F}(x) < 1\}, \quad E,F \in \Sigma, \tag{1}$$

is a pseudo-metric on Σ.

- $E \triangle F$ means a symmetrical difference of sets
- it is related to universal τ_T-submeasures
- If the condition

$$\gamma_E(x) = \varepsilon_0(x) \Rightarrow E = \emptyset, \quad x > 0,$$

is fulfilled, then χ given by (1) is a metric on Σ.

Pseudo-metrics and metrics generated by $\tau_{T,L}$-submeasures

Proposition

Let $L \in \mathcal{L}$ be an operation on $\overline{\mathbb{R}}_+$ such that $L \leq K_1$. If γ is a τ_{L,T_1}-submeasure on Σ and t is an additive generator of a continuous Archimedean t-norm T such that $T \leq T_1$, then

$$\mu_t(E,F) = \sup\{x > 0;\ t(\gamma_{E \triangle F}(x)) \geq x\}, \quad E,F \in \Sigma, \tag{2}$$

is a pseudo-metric on Σ.

- it is related to τ_T-submeasures where T has an additive generator
- If the condition

$$\gamma_E(x) = \varepsilon_0(x) \Rightarrow E = \emptyset, \quad x > 0,$$

is fulfilled, then μ_t given by (2) is a metric on Σ.

- Consider the *group* $\mathcal{H}$ *of automorphisms* of the unit interval $[0,1]$ acting on the class $\mathcal{B}$ of all functions from $[0,1]^2$ to $[0,1]$ as follows

$$(\Psi_h B)(x,y) = h^{-1}(B(h(x),h(y))), \quad h \in \mathcal{H},$$

for all $x, y \in [0,1]$.

- $\Psi_{\mathcal{H}}$ ... a class of all transformations (determined by a function $h \in \mathcal{H}$)
- indeed, $\Psi_{\mathcal{H}}$ is a group under the composition with the inverse $\Psi_h^{-1} = \Psi_{h^{-1}}$ and the identity $\Psi_{\mathrm{id}_{[0,1]}}$

Proposition

Let $h \in \mathcal{H}$. Then

(i) *if h is supermultiplicative, then for each $L_1, L_2 \in \mathcal{L}$ such that $L_1 \leq L_2$ holds $\Theta_{L_1,\Pi} \ll \Theta_{L_2,\Psi_h\Pi}$;*

(ii) *if the function $1-h(1-x)$ is subadditive, then for each $L_1, L_2 \in \mathcal{L}$ such that $L_1 \leq L_2$ holds $\Theta_{L_1,W} \ll \Theta_{L_2,\Psi_h W}$.*

Definition ($\tau_{L,A}$ -submeasure)

In order to generalize the concept of $\tau_{L,T}$ -submeasure let us introduce some notations

- A binary ***aggregation function*** $A : [0,1]^2 \to [0,1]$ is a non-decreasing function in both components with the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$.

  The class of all binary aggregation functions will be denoted by $\mathcal{A}$.

- for $(L, A) \in \mathcal{L} \times \mathcal{A}$ we have a function

$$\tau_{L,A}(G,H)(x) = \sup_{L(u,v)=x} A(G(u), H(v))$$

  left-continuity of A ensures that $\tau_{L,A}$ is a binary operation on Δ^+, however, $\tau_{L,A}$ need not be associative in general, but it has good properties on Δ^+

Definition

Let $(L, A) \in \mathcal{L} \times \mathcal{A}$ and Σ be a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ such that

(a") $\gamma_\emptyset(x) = \varepsilon_0(x), x > 0$;

(b") $\gamma_E(x) \geq \gamma_F(x), x > 0$ whenever $E \subset F$;

(c") $\gamma_{E \cup F}(L(x, y)) \geq A(\gamma_E(x), \gamma_F(y)),\ x, y > 0, E, F \in \Sigma$,

is said to be a $\tau_{L,A}$ -*submeasure*.

- **Example:** Let η be a numerical submeasure on Σ and $E \in \Sigma$. Then $\gamma \in \Theta_{C_\lambda^{GH}}$, where

$$\gamma_E(x) = \exp\left(-\left[\max\{\eta(E) - x, 0\}\right]^{1/\lambda}\right)$$

corresponds to the *Gumbel-Hougaard family* of (strict) copulas C_λ^{GH}, $\lambda \in [1, +\infty[$.

Definition

Let $(L, A) \in \mathcal{L} \times \mathcal{A}$ and Σ be a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ such that

(a") $\gamma_\emptyset(x) = \varepsilon_0(x), x > 0$;

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is said to be a $\tau_{L,A}$ -*submeasure*.

- **Example:** Let η be a numerical submeasure on Σ and $E \in \Sigma$. Then $\gamma \in \Theta_{\mathbf{M}_p}$, where

$$\gamma_E(x) = 2^{-1/p}\left(1 + \left(\max\left\{\min\left\{\sqrt[p]{\max\{1 + x - \eta(E), 0\}}, 1\right\}, 0\right\}\right)^p\right)^{1/p}$$

corresponds to the *Hölder mean* $\mathbf{M}_p(x, y) := \left(\frac{x^p + y^p}{2}\right)^{1/p}, p > 0$.

Lattice structure of submeasure spaces in $\Theta_{\mathcal{S}}$

If we denote set of all copulas, semi-copulas, quasi-copulas $\mathcal{C}, \mathcal{Q}, \mathcal{S}$, then

$$\mathcal{C} \subset \mathcal{Q} \subset \mathcal{S}.$$

Semi-copula is an aggregation function $S : [0,1]^2 \to [0,1]$ with 1 as its neutral element.

- every semi-copula may be represented as the point-wise supremum ($\vee$) and infimum ($\wedge$) of a suitable subset of t-norms
- Observe that if γ is a τ_{S_1}- and τ_{S_2}-submeasure for some $S_1, S_2 \in \mathcal{S}$, then γ is a $\tau_{S_1 \vee S_2}$- as well as $\tau_{S_1 \wedge S_2}$-submeasure.

Lattice structure of submeasure spaces in $\Theta_{\mathcal{S}}$

- for $S_1, S_2 \in \mathcal{S}$ put

$$\Theta_{S_1} \sqcup \Theta_{S_2} = \Theta_{S_1 \wedge S_2} \quad \text{and} \quad \Theta_{S_1} \sqcap \Theta_{S_2} = \Theta_{S_1 \vee S_2}$$

  It is easy to see that $\sqcup$ and $\sqcap$ are *lattice operations*.
- since $(\mathcal{S}, \leq, \vee, \wedge)$ is a complete lattice, then we have the following observation:

Proposition

The family $\Theta_{\mathcal{S}}$ of all probabilistic submeasure spaces is a distributive lattice.

Since for each $S \in \mathcal{S}$ holds $\Theta_M \ll \Theta_S \ll \Theta_D$, thus $\Theta_{\mathcal{S}}$ is a *bounded distributive lattice*.

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