Fuzzy Orders for Solving MOLP Problems

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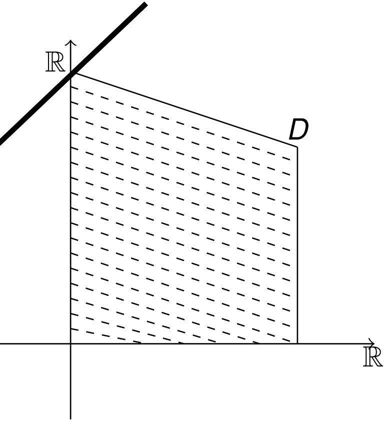
Problem formulation

In our work we observe a multi-objective linear programming problem, which can be represented as follows:
$MAX\ Z$, where $Z = (z_1, ..., z_k)$ is a vector of objectives,
$z_i = \sum_{j=1}^{n} c_{ij} x_j$ where $i = 1, .., k$,
subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $i = 1, ..., m$.
That is we should find a vector $x^o = (x_1^o, ..., x_n^o)$ which maximizes k objective functions with n variables, and m constraints.



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subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$,

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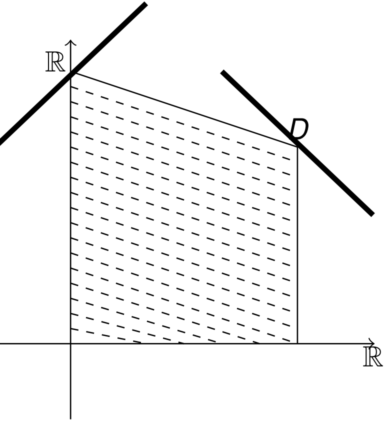

$MAX\ (z_1, z_2)$

$z_i = \sum_{j=1}^{n} c_{ij} x_j$ where $i = 1, 2$

subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$,

$i = 1, ..., m$.

Fuzzy approach

Membership functions:

$$\mu_i(x) = \begin{cases} 0, & z_i(x) < z_i^{min}, \\ \dfrac{z_i(x) - z_i^{min}}{z_i^{max} - z_i^{min}}, & z_i^{min} \leq z_i(x) \leq z_i^{max}, \\ 1, & z_i(x) > z_i^{max}. \end{cases}$$

$$\max_{x} \min_{i} \mu_i(x)$$

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$x \doteq y \Leftrightarrow z(x) = z(y)$
$x \preceq y \Leftrightarrow z(x) \leq z(y)$

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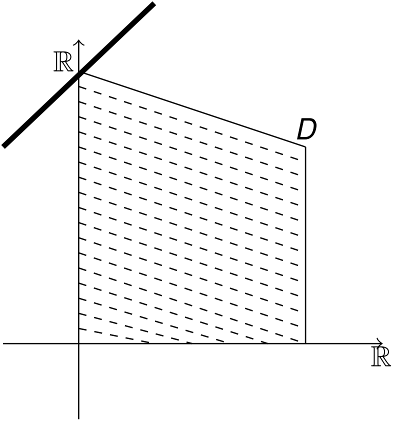

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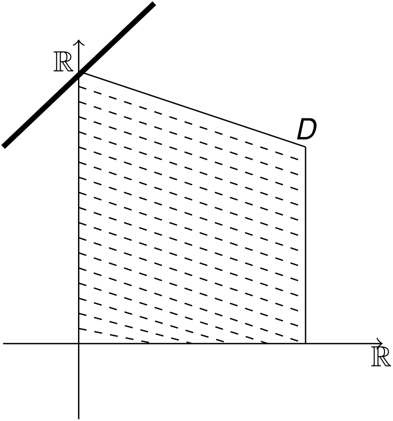

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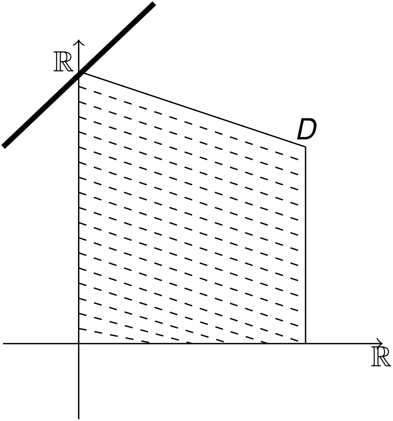

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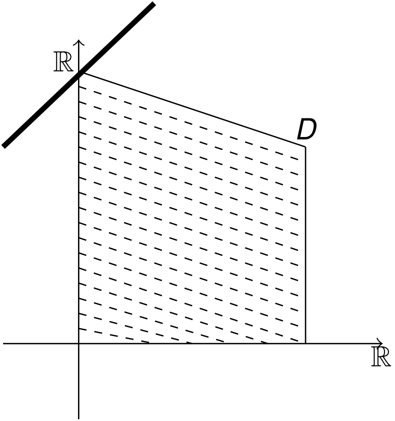

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Fuzzy order approach

1. We define fuzzy order relations P_i which generalize the following crisp order relations $x \preceq_i y \Leftrightarrow z_i(x) \leq z_i(y)$, $i = 1, ..., k$. Thus each fuzzy order relation describes corresponding objective function z_i.
2. We aggregate fuzzy orders using an aggregation function A which preserves the properties of initial fuzzy orders.

$$P(x, y) = A(P_1(x, y), ..., P_k(x, y)).$$

   Thus the aggregated fuzzy order relation P provides the information about all objective functions.
3. We maximize aggregated fuzzy order relation.

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Definition of a fuzzy equivalence relation

Definition

A fuzzy binary relation $E : X \times X \to [0, 1]$ on a set X is called fuzzy equivalence relation with respect to a t-norm T, for brevity T-equivalence, if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

1. $E(x, x) = 1$ reflexivity;
2. $E(x, y) = E(y, x)$ symmetry;
3. $T(E(x, y), E(y, z)) \leq E(x, z)$ T-transitivity.

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Fuzzy equivalence relation

Theorem

Let T be a continuous Archimedean t-norm with an additive generator t. For any pseudo-metric d, the mapping

$$E_d(x,y) = t^{(-1)}(\min(d(x,y), t(0)))$$

is a T-equivalence.

Example

Let us consider the set of real numbers $X = \mathbb{R}$ and metric $d(x,y) = |x-y|$ on it. Taking into account that $t_L(x) = 1 - x$ is an additive generator of T_L (Łukasiewicz t-norm) and that $t_P(x) = -ln(x)$ is an additive generator of T_P (product t-norm), we obtain two fuzzy equivalence relations:

$$E_L(x,y) = \max(1 - |x-y|, 0);$$

$$E_P(x,y) = e^{-|x-y|}.$$

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Let X be a set. By a fuzzy order relation with respect to a t-norm T and T-equivalence E we call a fuzzy relation $P : X \times X \to [0, 1]$ such that

1. $P(x, x) = 1 \ \forall x \in X$ reflexivity;
2. $T(P(x, y), P(y, z)) \leq P(x, z) \ \forall x, y, z \in X$ T-transitivity;
3. $T(P(x, y), P(y, x)) \leq E(x, y) \ \forall x, y \in X$ T-E-antisymmetry.

A pair (X, P) is called a fuzzy ordered set.
A fuzzy ordering P is called strongly linear if and only if

$$\forall x, y \in X : \quad \max(P(x, y), P(y, x)) = 1.$$

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Construction of fuzzy orderings (U.Bodenhofer)

Let X be a set.

Linear ordering $\leq$ **+** T-equivalence E (compatible with $\leq$) **=**

$$P(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ E(x,y), & \text{otherwise} \end{cases}$$

- a strongly linear T-E-ordering on X.

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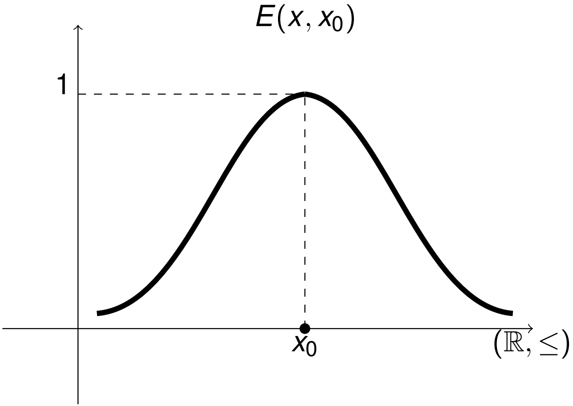

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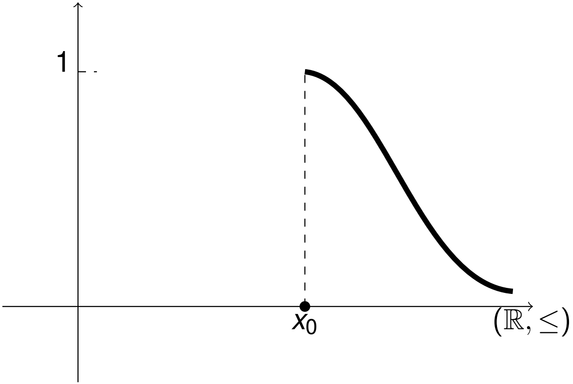

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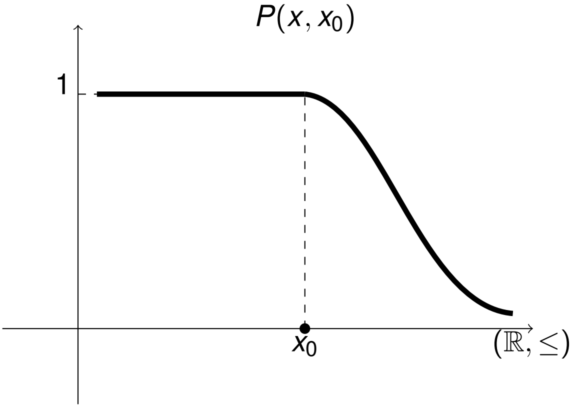

1. $d_i(x,y) = \frac{|z_i(x)-z_i(y)|}{z_i^{max}-z_i^{min}}$ are pseudo-metrics
2. $E_i(x,y) = t^{(-1)}(\min(\frac{|z_i(x)-z_i(y)|}{z_i^{max}-z_i^{min}}, t(0)))$ are fuzzy T-equivalence relations
3. we build fuzzy order relations (T-E-orders) by the following way:

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Aggregation of fuzzy relations

We aggregate fuzzy orders P_i $i \in \{1, ..., k\}$ using an aggregation function A which preserves the properties of initial fuzzy orders:

$$P(x, y) = A(P_1(x, y), ..., P_k(x, y))$$

Definition

Consider an n-argument aggregation function $A^n : [0, 1]^n \to [0, 1]$ and an m-argument aggregation function $B^m : [0, 1]^m \to [0, 1]$. We say that A^n dominates B^m if for all $x_{i,j} \in [0, 1]$ with $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$ the following property holds:

$$B^m(A^n(x_{1,1}, ..., x_{1,n}), ..., A^n(x_{m,1}, ..., x_{m,n})) \leq$$

$$\leq A^n(B^m(x_{1,1}, ..., x_{m,1}), ..., B^m(x_{1,n}, ..., x_{m,n})).$$

Theorem

Let $|X| > 3$ and let T be a t-norm. An aggregation function A preserves T-transitivity of fuzzy relations on X if and only if A belongs to the class of aggregation functions which dominate T.

Example

For any $k > 2$ and any $p = (p_1, ..., p_k)$ with $\sum_{i=1}^{k} p_i \geq 1$ and $p_i \in [0, \infty]$ k-ary aggregation function

$$A_p(x_1, ..., x_k) = \prod_{i=1}^{k} x_i^{p_i}$$

dominates the product t-norm T_P.

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$$A_p(x_1, ..., x_k) = \max(\sum_{i=1}^{k} x_i \cdot p_i + 1 - \sum_{i=1}^{k} p_i, 0)$$

dominates the Łukasiewicz t-norm T_L.

Theorem

Let $|X| > 3$ *and let* T *be a t-norm. If* E_i *for all* $i \in \{1, ..., n\}$ *are fuzzy equivalence relations (*T*-equivalences) then*

$$E(x, y) = A(E_1(x, y), ..., E_n(x, y))$$

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Theorem

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Further the multi-objective linear programming problem comes to the following problem:

$$\max_{y} \min_{x} P(x, y) \qquad (P)$$

Definition

x^* is called Pareto optimal solution if and only if there does not exist another $x \in D$ such that $z_i(x) \leq z_i(x^*)$ for all i and $z_j(x) \neq z_j(x^*)$ for at least one j.

Theorem

An optimal solution x^* to the problem (P) is a Pareto optimal solution if it is unique optimal solution.

Theorem

An optimal solution x^* to the problem (P) is a Pareto optimal solution if for all x and y $z_i(x) > z_i(y) \Rightarrow P_i(x, y) < 1$, A is a strictly monotone function and set D is linearly connected.

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$$\max_{y} \min_{x} P(x, y) \qquad (P)$$

Definition

x^* is called Pareto optimal solution if and only if there does not exist another $x \in D$ such that $z_i(x) \leq z_i(x^*)$ for all i and $z_j(x) \neq z_j(x^*)$ for at least one j.

Theorem

An optimal solution x^* to the problem (P) is a Pareto optimal solution if it is unique optimal solution.

Theorem

An optimal solution x^* to the problem (P) is a Pareto optimal solution if for all x and y $z_i(x) > z_i(y) \Rightarrow P_i(x, y) < 1$, A is a strictly monotone function and set D is linearly connected.

Numerical example

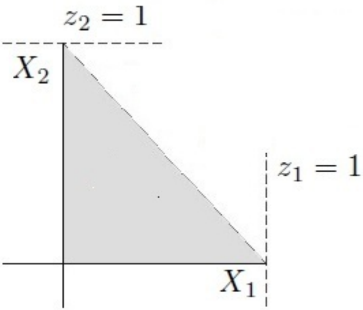
Let us observe the following linear programming problem:

$\max z_1 = x_1,$

$\max z_2 = x_2,$

s.t. $x_1 + x_2 \leq 1,$

$x_1, x_2 \geq 0.$



Numerical example

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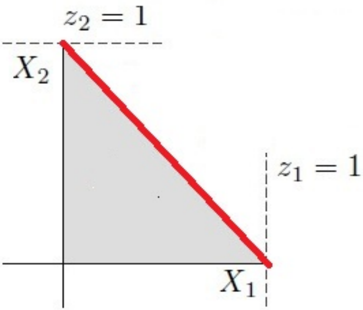

Numerical example(Łukasiewicz t-norm)

We solve the following problem: $\max_{y \in D} \min_{x \in D} P(x, y)$.

$$f(y) = \min_{x \in B} P(x, y) :$$

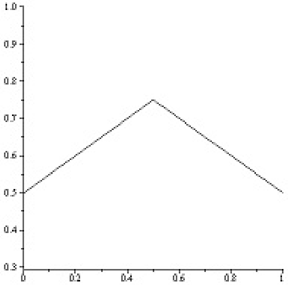

Figure:

$$P(x, y) = \frac{P_1(x,y)+P_2(x,y)}{2}$$

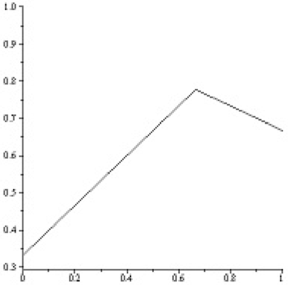

Figure:

$$P(x, y) = \frac{2P_1(x,y)+P_2(x,y)}{3}$$

Numerical example(Product t-norm)

We solve the following problem: $\max_{y \in D} \min_{x \in D} P(x, y)$.

$$f(y) = \min_{x \in B} P(x, y) :$$

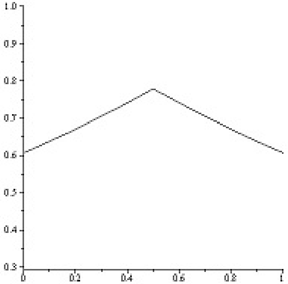

Figure:

$$P(x, y) = P_1(x, y)^{\frac{1}{2}} \cdot P_2(x, y)^{\frac{1}{2}}$$

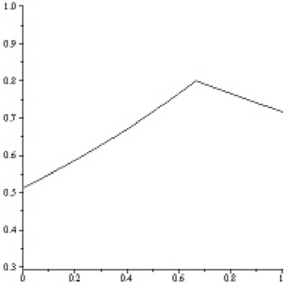

Figure:

$$P(x, y) = P_1(x, y)^{\frac{2}{3}} \cdot P_2(x, y)^{\frac{1}{3}}$$

Thank you for attention!