Robust Integrals

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Outline of talk

Choosing from a set of alternatives

In many decision problems, several *alternatives*

$$A = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots\}$$

are evaluated with respect to a set of *criteria*

$$N = \{1, \cdots, n\}.$$

- We could evaluate a car with respect to criteria such as

  {maximum speed, price, acceleration, fuel consumption..};
- We could evaluate a students with respect to the notes on different subjects such as

  {Mathematics, Physics, Literature,...}.

Considering redundancy or synergy among criteria

The importance of a set of criteria is not necessarily the sum of the importance of each criterion in the set.

Situation of **redundancy**

- maximum speed and acceleration in evaluating cars
- Mathematics and Physics in evaluating a student

Situation o **synergy**

- maximum speed and price in evaluating cars
- Mathematics and Literature in evaluating a student

Thus, in order to express a decision, such as a choice from a given set of cars or a ranking of a set of students, it is necessary to choose how **to aggregate the evaluations** on considered criteria.

Non-additive integrals

If on each criterion a given alternative $\boldsymbol{x}$ is evaluated on the same scale (α, β), thus this alternative can be identified with a score vector

$$\boldsymbol{x} = (x_1, \ldots, x_n) \in (\alpha, \beta)^n$$

where x_i is the evaluation of $\boldsymbol{x}$ with respect to the i^{th} criterion. In order to aggregate these evaluations, several non additive integrals have been introduced in the last sixty years. Among them, we remember

- the Choquet integral (Choquet (1953)
- the Shilkret integral (Shilkret (1971))
- the Sugeno integral (Sugeno (1974))

The Choquet integral

Definition

A capacity is function $\mu : 2^N \rightarrow [0,1]$ *satisfying:*

1. $\mu(\emptyset) = 0, \ \mu(N) = 1,$
2. *for all* $A \subseteq B \subseteq N, \ \mu(A) \leq \mu(B).$

Definition

The Choquet integral (Choquet (1953)) of a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in (\alpha, \beta)^n \subseteq [0, +\infty\,[\,^n$ *with respect to the capacity* μ *is given by*

$$Ch(\boldsymbol{x}, \mu) = \int_0^{\infty} \mu\left(\{i \in N : x_i \geq t\}\right) dt. \qquad (1)$$

Schmeidler (Schmeidler (1986)) extended the above definition to negative values too, moreover he characterized the Choquet integral in terms of monotonicity and comonotonic additivity.

Definition

The Choquet integral of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in (\alpha, \beta)^n$ *with respect to the capacity* μ *is given by*

$$Ch(\mathbf{x}, \mu) = \int_{-\infty}^{0} [1 - \mu(\{i \in N : x_i \geq t\})] \, dt + \int_{0}^{\infty} \mu(\{i \in N : x_i \geq t\}) \, dt \quad (2)$$

alternatively written

$$Ch(\mathbf{x}, \mu) = \int_{\min_i x_i}^{\max_i x_i} \mu(\{i \in N : x_i \geq t\}) \, dt + \min_i x_i \quad (3)$$

Interval evaluations on each criterion

Suppose that, for a given alternative $\boldsymbol{x}$ we have, on each criterion

$$i \in N = \{1, \cdots, n\}$$

the knowledge of an interval containing the exact evaluation

$$[\underline{x}_i, \overline{x}_i]$$

Thus, the alternative $\boldsymbol{x}$ can be identified with a (score) vector

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_i, \overline{x}_i], \ldots, [\underline{x}_n, \overline{x}_n]) \in I^n_{[a,b]} \tag{4}$$

being

$$I_{[a,b]} = \{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}, \ \underline{x} \leq \overline{x}\}$$

Pessimistic and optimistic evaluation of $\boldsymbol{x}$

We associate to every alternative $\boldsymbol{x} \in I^n_{[a,b]}$ the vector of all the worst (or pessimistic) evaluations on each criterion

$$\underline{\boldsymbol{x}} = (\underline{x}_1, \ldots, \underline{x}_n)$$

and the vector of all the best (or optimistic) evaluations on each criterion

$$\overline{\boldsymbol{x}} = (\overline{x}_1, \ldots, \overline{x}_n)\,.$$

The elements of $I^n_{[a,b]}$ will be, indifferently, called alternatives or, simply, vectors.

Interval capacity (a)

Let us consider the set

$$\mathcal{Q} = \{(A, B) \mid A \subseteq B \subseteq N\}$$

On $\mathcal{Q}$ we define a binary relation

$$(A, B) \precsim (C, D) \quad \Leftrightarrow \quad A \subseteq C \quad \text{and} \quad B \subseteq D \tag{5}$$

with respect to $\precsim$, $\mathcal{Q}$ is a lattice, where

$$\sup\{(A, B), (C, D)\} = (A \cup C, B \cup D)$$

and

$$\inf\{(A, B), (C, D)\} = (A \cap C, B \cap D).$$

Interval capacity (b)

Regarding the significance of $\mathcal{Q}$ in this work, let us consider the alternative

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$$

and a fixed **evaluation level** t.

Aggregating the criteria whose pessimistic evaluation of $\boldsymbol{x}$ is at least t

$$A_t = \{i \in N \mid \underline{x}_i \geq t\}$$

and the criteria whose optimistic evaluation of $\boldsymbol{x}$ is at least t

$$B_t = \{i \in N \mid \overline{x}_i \geq t\}$$

thus,

$$A_t \subseteq B_t \Rightarrow (A_t, B_t) \in \mathcal{Q}$$

Interval capacity (c)

Definition

A function $\mu_r : \mathcal{Q} \rightarrow [0,1]$ is an interval capacity on $\mathcal{Q}$ if

- $\mu_r(\emptyset, \emptyset) = 0$;
- $\mu_r(N, N) = 1$;
- $\mu_r(A, B) \leq \mu_r(C, D)$ *whenever* $(A, B) \precsim (C, D)$.

An interval capacity is an useful tool to assign a “weight” to the elements

$$(A_t, B_t) = (\{i \in N \mid \underline{x}_i \geq t\}, \{i \in N \mid \overline{x}_i \geq t\}) \in \mathcal{Q}$$

The Robust Choquet Integral (RCI)

Definition

The Robust Choquet Integral (RCI) of a vector

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \in I^n_{[a,b]}$$

with respect to an interval capacity $\mu_r : 2^N \rightarrow [0,1]$ *is:*

$$\begin{aligned} Ch_r(\boldsymbol{x}, \mu_r) \quad =: \quad & \int_{\min\{\underline{x}_1,\ldots,\underline{x}_n\}}^{\max\{\overline{x}_1,\ldots,\overline{x}_n\}} \mu_r(\{i \in N \mid \underline{x}_i \geq t\}, \{i \in N \mid \overline{x}_i \geq t\})dt \\ & + \min\{\underline{x}_1, \ldots, \underline{x}_n\}. \end{aligned} \tag{6}$$

Note that being in the (6) the integrand bounded and not increasing, the integral is the standard Riemann integral.

Relation with the Choquet integral (a)

Givien an interval capacity

$$\mu_r : \mathcal{Q} \to [0, 1]$$

a capacity $\nu : 2^N \to [0, 1]$ is defined by setting

$$\nu(A) = \mu_r(A, A) : 2^N \to [0, 1], \qquad \text{for all } A \subseteq N \tag{7}$$

Due to the *monotonicity* of the RCI

$$Ch_r(\underline{\boldsymbol{x}}, \mu_r) = Ch(\underline{\boldsymbol{x}}, \nu) \leq Ch_r(\boldsymbol{x}, \mu_r) \leq Ch(\overline{\boldsymbol{x}}, \nu) = Ch_r(\overline{\boldsymbol{x}}, \mu_r). \tag{8}$$

Note that when on each criterion we have exact evaluations, $\underline{x}_i = \overline{x}_i$,

$$Ch_r(\boldsymbol{x}, \mu_r) = Ch(\boldsymbol{x}, \nu).$$

the RCI of $\boldsymbol{x}$ w.r.t. μ_r collapses on the Choquet integral of $\boldsymbol{x}$ w.r.t. ν.

Relation with the Choquet integral (b)

On the other hand, starting from two capacities

$$\underline{\nu} : 2^N \rightarrow [0,1]$$

$$\overline{\nu} : 2^N \rightarrow [0,1]$$

for all $\alpha \in (0,1)$, a *separable* interval capacity is defined by means of

$$\mu_r(A,B) = \alpha\underline{\nu}(A) + (1-\alpha)\overline{\nu}(B), \qquad \text{for all } (A,B) \in \mathcal{Q} \tag{9}$$

In this case

$$Ch_r(\boldsymbol{x}, \mu_r) = \alpha Ch(\underline{\mathbf{x}}, \underline{\nu}) + (1-\alpha) Ch(\overline{\boldsymbol{x}}, \overline{\nu}) \tag{10}$$

For example, given a capacity ν, one could think to obtain a lower, an intermediate and an upper aggregate evaluation of an alternative $\boldsymbol{x}$

$$Ch(\underline{\mathbf{x}}, \nu) \leq \alpha Ch(\underline{\mathbf{x}}, \nu) + (1-\alpha) Ch(\overline{\boldsymbol{x}}, \nu) \leq Ch(\overline{\boldsymbol{x}}, \nu). \tag{11}$$

Clearly, our approach is more general.

An illustrative example

Example

Taking inspiration from an example very well known in the specialized literature, Grabisch (1996), let us consider a case of evaluation of three students in Mathematics, Physics and Literature.

- The students are evaluated on each subject by a 10 point scale,
- some evaluation are imprecise (typical situation in the middle of a school year),
- the dean of the school ranks the students as follows:

$$S_2 \succ S_1 \succ S_3.$$

	Mathematics	Physics	Literature
S_1	8	8	7
S_2	$[7, 8]$	8	$[6, 8]$
S_3	9	9	$[5, 6]$

Table: Students' evaluations

Dean ranking: $S_2 \succ S_1 \succ S_3$.

The rationale of this ranking is that:

- $S_1 \succ S_3$: the better evaluations of S_3 in scientific subjects are redundant, the dean retains relevant the better evaluation of S_1 in Literature, where S_3 risks an insufficiency. In other words, when the scientific evaluation is fairly high, Literature becomes very important;
- $S_2 \succ S_1$ the conjoint evaluation in Mathematics and Physics is very similar, also considering the redundancy of the two subjects. However S_2 has the same average in Literature but a greater potential;
- $S_2 \succ S_3$ by transitivity of preferences.

	Mathematics	Physics	Literature
S_1	8	8	7
S_2	7	8	6
S_3	9	9	5

Table: pessimistic evaluations S_1 dominates S_2, S_3 has the best average

	Mathematics	Physics	Literature
S_1	8	8	7
S_2	8	8	8
S_3	9	9	6

Table: Optimistic evaluations: S_2 dominates S_1, S_3 has the best average.

The RCI permits to represent the preferences of the dean. Let

$$N = \{M,\ Ph,\ L\}$$

be set of criteria and let us identify the three students respectively with the three vectors:

$$\begin{aligned} \boldsymbol{x}_1 &= ([8,8],[8,8],[7,7]) \\ \boldsymbol{x}_2 &= ([7,8],[8,8],[6,8]) \\ \boldsymbol{x}_3 &= ([9,9],[9,9],[5,6])\,. \end{aligned}$$

The RCI represents the preferences of the dean if there exists an interval capacity μ_r such that

$$Ch_r(\boldsymbol{x}_2,\mu_r) > Ch_r(\boldsymbol{x}_1,\mu_r) > Ch_r(\boldsymbol{x}_3,\mu_r),$$

that is

$$6+\mu_r(\{M,Ph\},N)+\mu_r(\{Ph\},N)>7+\mu_r(\{M,Ph\},\{M,Ph\})>$$

$$>5+\mu_r(\{M,Ph\},\mathcal{S})+3\mu_r(\{M,Ph\},\{M,Ph\}).$$

Which can be explained, for example, by setting

$$\begin{cases}\mu_r(\{M,Ph\},N)=0.9\\ \mu_r(\{Ph\},N)=0.7\\ \mu_r(\{M,Ph\},\{M,Ph\})=0.5\end{cases}$$

Integral characterization

The RCI is a function aggregating interval evaluations in a single number. In order to get an axiomatic characterization we need to extend the notions of

- additivity,
- monotonicity,
- co-monotonicity

Definition

For every $\alpha, \beta \in \mathbb{R}$ *and* $[x_1, y_1], [x_2, y_2] \in I_{[a,b]}$ *we define the following interval mixture operation:*

$$\alpha \cdot [x_1, y_1] + \beta \cdot [x_2, y_2] = [\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2].$$

Thus, for all vectors (alternative) $\boldsymbol{x}, \boldsymbol{y} \in I^n_{[a,b]}$ and for all $\alpha, \beta \in \mathbb{R}$,

$$(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) \in I^n_{[a,b]}$$

is the vector defined by

$$(\alpha x + \beta y)_i = \alpha x_i + \beta y_i \qquad \text{for all } i \in N$$

Definition

We define $[\alpha, \beta] \leq [\alpha_1, \beta_1]$ *whenever* $\alpha \leq \alpha_1$ *and* $\beta \leq \beta_1$.

Remark

$(I_{[a,b]}, \leq)$ *is a partial ordered set, not complete, e.g. we are not able to establish the preference between* $[2, 5]$ *and* $[3, 4]$.

$$\mathbf{x} \leq \mathbf{y} \text{ means } x_i \leq y_i \text{ for all } i \in N.$$

Definition

$$\boldsymbol{x} = \left([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]\right)$$

$$\boldsymbol{y} = \left([\underline{y}_1, \overline{y}_1], \ldots, [\underline{y}_n, \overline{y}_n]\right)$$

are comonotonic (or comonotone) if they are, in $\mathbb{R}^{2n}$,

$$\boldsymbol{x}^* = (\underline{x}_1, \ldots, \underline{x}_n, \ldots, \overline{x}_1, \ldots, \overline{x}_n)$$

$$\boldsymbol{y}^* = \left(\underline{y}_i, \ldots, \underline{y}_n, \ldots, \overline{y}_1, \ldots, \overline{y}_n\right)$$

Remark

if $\mathbf{x}$ *and* $\mathbf{y}$ *are co-monotonic, then both* $\underline{\mathbf{x}}$ *and* $\underline{\mathbf{y}}$ *are co-monotonic as well as* $\overline{\mathbf{x}}$ *and* $\overline{\mathbf{y}}$ *are co-monotonic. The reverse is generally false, for example if* $N = \{1, 2\}$, $\mathbf{x} = ([1, 3], [2, 4])$ *and* $\mathbf{y} = ([1, 3], [4, 5])$ *are non co-monotonic, although* $\underline{\mathbf{x}}$ *is co-monotonic with* $\underline{\mathbf{y}}$ *and* $\overline{\mathbf{x}}$ *is co-monotonic with* $\overline{\mathbf{y}}$.

For all $(A, B) \in \mathcal{Q}$ we define a generalized indicator function

$$1_{(A,B)} : N \to \{[0,0],[0,1],[1,1]\}$$

by means of

$$1_{(A,B)}(i) = \begin{cases} [1,1] = 1 & i \in A \\ [0,1] & i \in B \setminus A \\ [0,0] = 0 & i \in N \setminus B \end{cases} \tag{12}$$

Clearly, if $A = B$,

$$1_{(A,A)} = 1_A$$

For any interval capacity μ_r, by definition :

$$Ch_r(\mathbf{1}_{(A,B)}, \mu_r) = \mu_r(A, B)$$

Properties of the RCI

- **Idempotency.** For all $\mathbf{k} = (k, k, \dots, k)$ with $k \in \mathbb{R}$,

$$Ch_r(\mathbf{k}, \mu_r) = k$$

- **Positive homogeneity.** Let $a > 0$ and $\boldsymbol{x} \in I^n_{[a,b]}$,

$$Ch_r(a \cdot \boldsymbol{x}, \mu_r) = a \cdot Ch_r(\boldsymbol{x}, \mu_r)$$

- **Monotonicity.** Let $\boldsymbol{x}$, $\mathbf{y} \in I^n_{[a,b]}$ with $\boldsymbol{x} \leq \mathbf{y}$,

$$Ch_r(\boldsymbol{x}, \mu_r) \leq Ch_r(\mathbf{y}, \mu_r)$$

- **Co-monotonic additivity.** if $\boldsymbol{x}$, $\mathbf{y} \in I^n_{[a,b]}$ are co-monotonic,

$$Ch_r(\boldsymbol{x} + \mathbf{y}, \mu_r) = Ch_r(\boldsymbol{x}, \mu_r) + Ch_r(\mathbf{y}, \mu_r)$$

The above properties are characterizing for the RCI.

Theorem

Let $G : I^n_{[a,b]} \to \mathbb{R}$ be a an (generalized) aggregation function satisfying

- $G(\mathbf{1}_{(N,N)}) = 1$
- *(P3) Monotonicity*
- *(P4) Co-monotonic additivity*

thus, by setting

$$\mu_r(A,B) = G\left(\mathbf{1}_{(A,B)}\right) \text{ for all } (A,B) \in \mathcal{Q}$$

it follows that:

$$G(\mathbf{x},\mu_r) = Ch_r(\mathbf{x},\mu_r), \quad \text{for all } \mathbf{x} \in I^n_{[a,b]}$$

Generalizing the concept of interval to m-points interval

Oztürk *et al.* (2011) generalized the concept of interval (allowing the presence of more than two points). Image that on each criterion an alternative $\boldsymbol{x}$ is evaluated m times.

$$\boldsymbol{x} = (x_1, \ldots, x_n)$$

being for all $i = 1, \ldots, n$ and for all $j = 1, \ldots, m-1$

$$x_i = (f_1(x_i), \ldots f_m(x_i)), \qquad f_j(x_i) \leq f_{j+1}(x_i)$$

E.g. m=3 corresponds to have on each criterion a pessimistic, a realistic and an optimistic evaluation.

Let us define

$$\mathcal{Q}_m = \{(A_1, \ldots, A_m) \mid A_1 \subseteq A_2 \ldots \subseteq A_m \subseteq N\}$$

Definition

An m-interval capacity is a function $\mu_m : \mathcal{Q}_m \to [0, 1]$ such that

- $\mu_m(\emptyset, \ldots, \emptyset) = 0$
- $\mu_m(N, \ldots, N) = 1$
- $\mu_m(A_1, \ldots, A_m) \leq \mu_m(B_1, \ldots, B_m)$*, with* $A_i \subseteq B_i \subseteq N, \ \forall i = 1, \ldots, m$

Definition

The Robust Choquet Integral of $\mathbf{x}$*(m-points interval valued) w.r.t. the m-interval capacity* μ_m *is*

$$\int_{\min_i f_1(x_i)}^{\max_i f_m(x_i)} \mu_m\left(\{j \in N \mid f_1(x_j) \geq t\}, \ldots, \{j \in N \mid f_m(x_j) \geq t\} dt\right) + \min_i f_1(x_i)$$

The RCI and Möbius inverse

The following proposition gives the closed formula of the Möbius inverse of a function on $\mathcal{Q}$.

Proposition

Suppose $f, g : \mathcal{Q} \to \mathbb{R}$ are two real valued functions on $\mathcal{Q}$. Then

$$f(A, B) = \sum_{(C,D) \precsim (A,B)} g(C, D) \tag{13}$$

if and only if

$$g(A, B) = \sum_{\emptyset \subseteq X \subseteq A} (-1)^{|X|} \sum_{(C,D) \precsim (A \setminus X, B \setminus X)} (-1)^{|B \setminus A| - |D \setminus C|} f(C, D) \tag{14}$$

Proposition

$\mu_r : \mathcal{Q} \rightarrow \mathbb{R}$ is an interval capacity if and only if its Möbius inverse $m : \mathcal{Q} \rightarrow \mathbb{R}$ satisfies:

1. $m(\emptyset, \emptyset) = 0$
2. $\sum_{(A,B)\in\mathcal{Q}} m(A, B) = 1$
3. $\sum_{\{a\}\subseteq C\subseteq A} \sum_{C\subseteq D\subseteq B} m(C, D) \geq 0, \; \forall a \in A \subseteq B \in 2^N$
4. $\sum_{\{b\}\subseteq D\subseteq B} \sum_{C\subseteq A\cap D} m(C, D) \geq 0, \; \forall b \in B \supseteq A \in 2^N$

Proposition

Let $\mu_r : \mathcal{Q} \rightarrow [0, 1]$ be an interval capacity and let $m : \mathcal{Q} \rightarrow [0, 1]$ be its Möbius inverse, then for all $\boldsymbol{x} \in \mathcal{F}$

$$Ch_r(\boldsymbol{x}, \mu_r) = \sum_{(A,B)\in\mathcal{Q}} \min\left(\min_{i\in A} \underline{x}_i, \min_{i\in B} \overline{x}_i\right) m(A, B) \qquad (15)$$

The Robust Sugeno Integral

Suppose that $\boldsymbol{x}$ is evaluated on the scale $[0,1]$ on each criterion,

$$\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in [0,1]^n$$

The Sugeno Integral (Sugeno (1974)) of $\boldsymbol{x}$ w.r.t. the capacity μ is

$$S(\boldsymbol{x}, \mu) = \max_{i \in N} \left\{ \min \left\{ x_i, \mu \left(j \in N \mid x_j \geq x_i \right) \right\} \right\} \tag{16}$$

$$S(\boldsymbol{x}, \mu) = \max_{A \subseteq N} \left\{ \min \left\{ \min_{i \in A} x_i, \mu(A) \right\} \right\} \tag{17}$$

In the case of imprecise interval evaluations, we suppose that

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \dots, [\underline{x}_n, \overline{x}_n]), \qquad [\underline{x}_1, \overline{x}_1] \subseteq [0,1]$$

Considering the 2n vector

$$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) = (\underline{x}_1, \ldots, \underline{x}_n, \overline{x}_1, \ldots, \overline{x}_n)$$

Definition

The robust Sugeno integral of $\boldsymbol{x}$ *w.r.t. the interval capacity* μ_r *is*

$$S_r(\boldsymbol{x}, \mu_r) = \max_{i \in \{1,\ldots,2n\}} \left\{ \min \left\{ x_i, \mu_r \left(\left\{ j \in N \mid \underline{x}_j \geq x_i \right\}, \left\{ j \in N \mid \overline{x}_j \geq x_i \right\} \right) \right\} \right\} \quad (18)$$

Or, equivalently

$$S_r(\boldsymbol{x}, \mu_r) = \max_{(A,B) \in \mathcal{Q}} \left\{ \min \left\{ \min_{i \in A} \underline{x}_i, \min_{i \in B-A} \overline{x}_i, \mu_r(A, B) \right\} \right\} \quad (19)$$

An applicative examples

Example

This example shows the equivalence of formulation (18) *and* (19).

Let us suppose that $N = \{1, 2\}$ and consider

$$\boldsymbol{x} = ([5, 9], [2, 4])$$

Let be given the following interval capacity on $\mathcal{Q}$:

$$\mu_r(\emptyset, \emptyset) = 0, \ \mu_r(\emptyset, 1) = 3, \ \mu_r(\emptyset, 2) = 2, \ \mu_r(\emptyset, 12) = 5, \ \mu_r(1, 1) = 4,$$

$$\mu_r(1, 12) = 6, \ \mu_r(2, 2) = 4, \ \mu_r(2, 12) = 7, \ \mu_r(12, 12) = 10$$

Both using the (18) as well as the (18),

$$S_r(\boldsymbol{x}, \mu_r) = 4$$

The Robust Shilkret integral

Suppose that $\boldsymbol{x}$ is evaluated on a nonnegative scale on each criterion,

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$$

The Shilkret integral (Shilkret (1971)) of $\boldsymbol{x}$ w.r.t. the capacity μ is

$$Sh(x,\mu) = \max_{A \subseteq N} \left\{ \min_{i \in A} x_i \cdot \mu(A) \right\}$$

In the case of imprecise interval evaluations, we suppose that

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]), \qquad [\underline{x}_1, \overline{x}_1] \subseteq \mathbb{R}^n_+$$

Definition

*The robust Shilkret integral of **x** w.r.t. the interval capacity μ_r is*

$$Sh_r(x,\mu_r) = \max_{(A,B) \in \mathcal{Q}} \left\{ \min\left(\min_{i \in A} \underline{x}_i, \min_{i \in B \setminus A} \overline{x}_i \right) \cdot \mu_r(A,B) \right\} \tag{20}$$

Other possible extension of robustness

- the robust concave integral, generalizing the concave integral (Lehrer (2009))
- the robust universal integral generalizing the universal integral (Klement *et al.* (2010))
- robust bipolar integrals
- robust integral w.r.t. a level dependent interval capacity

THANKS FOR YOUR ATTENTION

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