Foundations of aggregation functions theory

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A bare outline

- A critical discussion of the basics of aggregation functions theory
- A new theoretical framework for aggregation with collateral parameters
- A close observation of the notions of unanimity, idempotency and asymptotic idempotency.

Idempotency

Let $\mathbb{I} := [0,1]$, fix $n \in \mathbb{N}$ and consider mappings of the kind

$$\mathbf{x} \mapsto G_n(\mathbf{x}) \in \mathbb{R},$$

where $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{I}^n$.

- Any $x \in \mathbb{I}$ is called an *idempotent* element for G_n if

$$\delta_{G_n}(x) = x,$$

  where $\delta_{G_n}(x) := G_n(x, ..., x)$ is the *diagonal section* of G_n.

- We say that G_n is idempotent if any $x \in \mathbb{I}$ is idempotent.

Aggregation Function

A mapping G_n is called an n-ary *Aggregation Function* (AF) in $\mathbb{I}$ if

- it is non-decreasing monotone in its components
- the border elements of $\mathbb{I}$ are idempotent

Aggregation Function

- The first natural requirement means that if even one of the input values increases, the representative output value should reflect this increase, or at worst, stay constant.
- The second requirement, nearly unanimously accepted as indispensable, is a boundary condition and it means, for instance, that if all the input values are equal to the worst possible level, then it is natural to expect that their correspondent output value preserves the same situation.
- Note that the two axioms easily imply $G_n(\mathbb{I}^n) \subseteq \mathbb{I}$.

A special case

When $n = 1$, the convention

$$G_1 \equiv id_{\mathbb{I}},$$

where $id(x) = x$, is very often considered.

It means that the aggregation of a singleton is not a true fusion.

Extended Function or Aggregation Operator?

- Generally, we do not know *a priori* the number of variables we have to deal with.
- We need a mathematical object more flexible than an AF, able to work with any number of input values.
- In literature, we find two equivalent definitions, with different terminology.

Extended Aggregation Function

Let $\boldsymbol{G}$: $\bigcup_{n=1}^{\infty} \mathbb{I}^n \to \mathbb{R}$ and $G_n := \boldsymbol{G}|_{\mathbb{I}^n}$.

- We say that $\boldsymbol{G}$ is an *Extended Aggregation Function* when every G_n is an AF.

Aggregation Operator

Let $\boldsymbol{G} = \{G_n\}_{n\in\mathbb{N}}$ be a sequence of functions $G_n : \mathbb{I}^n \to \mathbb{R}$.

- We say that $\boldsymbol{G}$ is an *Aggregation Operator* (AGOP) when every G_n is an AF.

The unary AF

Why an unary AF must necessarily be the identity function?

- It excludes several types of operators which still deserve to be called aggregators
- It may set strong constraints to the structure of the operator up to incompatibility with certain properties

The unary AF ••

- $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of RVs with support in $\mathbb{I}$ and suppose that any associated df F_n is continuous.

$$F_n(\mathbf{x}) = \mathbb{P}(X_1 < x_1, X_2 < x_2, \ldots, X_n < x_n)$$

Then $\{F_n\}_{n\in\mathbb{N}}$ respects all the conditions of an AGOP except for $F_1 \equiv id_{\mathbb{I}}$.

- The class of strict AGOPs with $e = 0$ as neutral element is empty. Indeed, if such an AGOP $\boldsymbol{G}$ existed, we'd have

$$G_1(1) = G_2(1,0) < G_2(1,1),$$

which is clearly incompatible with $G_1(1) = id_{\mathbb{I}}(1) = 1$.

The border conditions

Let $\boldsymbol{G}$ be given by

$$G_n(\mathbf{x}) = \frac{\sum_{i=1}^{n} x_i}{1 + \sum_{i=1}^{n} x_i}.$$

Then $\boldsymbol{G}$ respects all the conditions of an AGOP except for the border condition $\delta_{G_n}(1) = 1$ for all $n \in \mathbb{N}$, which is satisfied in the weaker form

$$\lim_{n \to \infty} \delta_{G_n}(1) = 1.$$

The border conditions ••

Fixed any $c \in \,]0,1[$, let $\boldsymbol{G}$ be given by

$$G_n(\mathbf{x}) = \begin{cases} c \cdot \dfrac{\sum_{i=1}^{n} x_i}{1+\sum_{i=1}^{n} x_i}, & \text{if } n \text{ is even;} \\ \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} x_i, & \text{otherwise.} \end{cases}$$

Again, $\boldsymbol{G}$ respects all the conditions of an AGOP except for the border condition $\delta_{G_n}(1) = 1$ for all $n \in \mathbb{N}$, which is satisfied in the weaker form

$$\limsup_{n \to \infty} \delta_{G_n}(1) = 1.$$

How to modify the boundary conditions?

The idea is to transpose the idempotency at the boundary of any AF in a sort of weakened idempotency for the whole operator.

- First proposal

$$\delta_{G_n}(1) = 1 \text{ for all } n \in \mathbb{N} \to \lim_{n\to\infty} \delta_{G_n}(1) = 1$$

- Second proposal

$$\delta_{G_n}(1) = 1 \text{ for all } n \in \mathbb{N} \to \limsup_{n\to\infty} \delta_{G_n}(1) = 1.$$

A new notion of AGOP

We say that $\boldsymbol{G}$ is an AGOP when

- every G_n is non-decreasing in each component;
- the following border conditions are satisfied:

$$\liminf_{n\to\infty} \delta_{G_n}(0) = 0 \quad \text{and} \quad \limsup_{n\to\infty} \delta_{G_n}(1) = 1.$$

Notion of static or dynamic property

A property $\mathcal{P}$ for an AGOP **G** is called

- *static* or *at fixed arity*, when every G_n verifies $\mathcal{P}$.
- *dynamic* or *at unfixed arity*, when it links G_n and G_m for different indices $n, m \in \mathbb{N}$.

Typical properties at fixed arity

- Continuity $\longrightarrow$ every G_n is continuous;
- Symmetry $\longrightarrow$ every G_n is symmetric;
- Idempotency $\longrightarrow$ every G_n is idempotent.

Examples of properties at unfixed arity

- An AGOP **G** is *asymptotically idempotent* when

$$\lim_{n\to\infty} \delta_{G_n}(x) = x.$$

- An AGOP **G** is *self-identical* when, given any $\mathbf{x} \in \mathbb{I}^n$, we have

$$G_n(\mathbf{x}) = G_{n+1}(x_1, ..., x_n, G_n(\mathbf{x})) \quad \text{for all } n \in \mathbb{N}.$$

Neutrality at fixed arity

- Let $G_n : \mathbb{I}^n \to \mathbb{I}$ be an n-ary function. We say that $e \in \mathbb{I}$ is a *neutral element* for G_n if, fixed any $i \in \{1, ..., n\}$, then

$$G_n(\mathbf{x}) = x_i$$

for any $\mathbf{x} \in \mathbb{I}^n$ verifying $x_j = e$ for all $j \neq i$.

- We say that $e \in \mathbb{I}$ is a *static neutral element* for an AGOP **G** when e is a neutral element for every G_n.

Neutrality at unfixed arity

We say that $e \in \mathbb{I}$ is a *dynamic neutral element* for an AGOP **G** when, given any $i \in \{1, ..., n+1\}$ and any $(x_1, ..., x_{n+1}) \in \mathbb{I}^{n+1}$ such that $x_i = e$, then

$$G_{n+1}(x_1, ..., x_{n+1}) = G_n(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$

for all $n \in \mathbb{N}$.

Independence of the two versions

Let $\mathbf{C} = \{C_n\}_{n\in\mathbb{N}}$ be given by

$$C_n(\mathbf{x}) = \min\{1, x_2, \ldots, x_n\} - \min\{1 - x_1, x_2, \ldots, x_n\}.$$

Note that $\mathbf{C}$ is a non-symmetric n-copula, hence a classical AGOP.

- $e = 1$ is a static neutral element;
- $e = 1$ is not a dynamic neutral element, since, for instance,

$$C_3\Big(1, \frac{1}{2}, \frac{1}{2}\Big) = \frac{1}{2} \neq 0 = C_2\Big(\frac{1}{2}, \frac{1}{2}\Big).$$

Independence of the two versions ••

Let **G** be the AGOP given by

$$G_n(\mathbf{x}) = \frac{\sum_{i=1}^{n} x_i}{1 + \sum_{i=1}^{n} x_i}.$$

- $e = 0$ is a dynamic neutral element;
- $e = 0$ is not a static neutral element, since, for instance,

$$G_n(1, 0, \dots, 0) = G_1(1) = \frac{1}{2} \neq 1.$$

A q-property

Let $q \in \mathbb{N}$, with $q \geq 2$.

We say that $\mathcal{P}$ is a q-property for an AGOP $\mathbf{G}$, when it involves every mapping G_n, with $n \geq q$.

Examples

- Let **G** be the AGOP defined as

$$G_n(\mathbf{x}) = \begin{cases} \frac{1}{q}\sum_{i=1}^{n} x_i, & \text{if } n \le q; \\ \frac{1}{n}\sum_{i=1}^{n} x_i, & \text{otherwise.} \end{cases}$$

- **G** is a q-idempotent AGOP.

Examples

Let **G** be the classical AGOP defined as

$$G_n(\mathbf{x}) = \begin{cases} \max(x_1^2, ..., x_n^2), & \text{if } n > 1; \\ id_{\mathbb{I}}, & \text{otherwise.} \end{cases}$$

We have that $e = 0$

- is neither a static nor a dynamic neutral element;
- is not a 2-static neutral element, since, for instance,

$$G_n(1, 0, \ldots, 0) = G_1(1) = \frac{1}{2} \neq 1;$$

- is a 2-dynamic neutral element.

The basic idea

- input values $\xrightarrow{G_n}$ output value

$$(x_1, ..., x_n) \xrightarrow{G_n} y;$$

- each input value is associated with a set of collateral parameters called *attributes*, which influence the result of aggregation

$$\Big((x_1, d_1), (x_2, d_2), ..., (x_n, d_n)\Big) \xrightarrow{H_n} \widetilde{y}.$$

The role of the attributes in an application

A central authority must collect the votes of anonymous peers of a network about the *trust value* they express with respect to the behavior of a certain peer

- first situation: all the votes have the same weight;
- second situation: the *network distance* between any voter and the judged peer is taken into account, in the sense that the bigger is such distance, the smaller is the reliability of the input value.

Aggregation without or with attributes

- first situation: choose a *root-mean-power* AF of the kind

$$G_n(\mathbf{x}) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}},$$

  for some $p \in \mathbb{R} \setminus \{0\}$.

- second situation: any input value x_k is accompanied by the distance $d_k \in]0, \infty[$, for $k = 1, ..., n$.

  Problem: how can we maintain the structure of the previous AF, taking into account the negative influence of any distance?

A generalized root-mean-power

We can solve this problem with the following choice, called *generalized root-mean-power*

$$H_n((x_1, d_1), ..., (x_n, d_n)) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}},$$

where $p = -\sum_{i=1}^{n} d_i$. We can show that H_n is

- increasing with respect to any input value;
- decreasing with respect to any distance.

Single-attribute AF

Fix any real interval $\mathbb{J}$ as domain of the single attribute.

- A mapping $H_n : (\mathbb{I} \times \mathbb{J})^n \to \mathbb{I}$ is a single-attribute n-ary AF when
$$H_n((x_1, d_1), ..., (x_n, d_n))$$
is non-decreasing monotone with respect to any x_k, for $k = 1, ..., n$.
- We say that H_n is *negative* (*positive*) when H_n is non-increasing (non-decreasing) monotone with respect to any d_k, for $k = 1, ..., n$.

Weighted aggregation functions

For any weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{I}^n$ such that $\sum_{i=1}^n w_i = 1$, the weighted arithmetic mean function

$$WAM(\mathbf{x}) := \sum_{i=1}^{n} w_i x_i$$

and the ordered weighted arithmetic mean function

$$OWA(\mathbf{x}) := \sum_{i=1}^{n} w_i x_{(i)},$$

where $x_{(1)} \leq x_{(2)} \leq \ldots x_{(n)}$, are particular cases of positive, single-attribute AFs (here, $\mathbb{J} = \mathbb{I}$).

Double-attribute AF

Fix any pair of real intervals $\mathbb{J}_1, \mathbb{J}_2$ as domains of the two attributes.

- A mapping $H_n : (\mathbb{I} \times \mathbb{J}_1 \times \mathbb{J}_2)^n \to \mathbb{I}$ is a double-attribute n-ary AF when

$$H_n((x_1, d_1, q_1), \ldots, (x_n, d_n, q_n))$$

  is non-decreasing monotone with respect to any x_k, for $k = 1, ..., n$.

- We say that H_n is *negative* (*positive*) when H_n is non-increasing (non-decreasing) monotone with respect to any d_k and q_k, for $k = 1, ..., n$.

A not monotonic double-attribute AF

Consider the following binary AF

$$F_2((x_1, d_1, q_1), (x_2, d_2, q_2)) = \arg\min_y \left(d_1 |x_1 - y|^{q_1} + d_2 |x_1 - y|^{q_2} \right)$$

where $d_1, d_2 \in \mathbb{J}_1 =]0, \infty[$ and $q_1, q_2 \in \mathbb{J}_2 =]1, \infty[$.

It can be seen that such AF is not monotone with respect to any of the two attributes.

Single-attribute AGOP

Fix an arbitrary sequence of n-tuples of attributes:

$$n \to \mathbf{d}^{(n)} := \{d_1^{(n)}, d_2^{(n)}, \dots, d_n^{(n)}\}$$

A sequence $\boldsymbol{H} = \{H_n\}_{n \in \mathbb{N}}$ of single-attribute AFs is a S-AGOP if the following border conditions hold:

$$\liminf_{n \to \infty} H_n((0, d_1^{(n)}), \dots, (0, d_n^{(n)})) = 0$$

and

$$\limsup_{n \to \infty} H_n((1, d_1^{(n)}), \dots, (1, d_n^{(n)})) = 1.$$

Weighted aggregation operators

Fix an arbitrary sequence of n-tuples of weights

$$n \to \mathbf{w}^{(n)} := \{w_1^{(n)}, w_2^{(n)}, ..., w_n^{(n)}\} \in \mathbb{I}^n$$

such that $\sum_{i=1}^{n} w_i^{(n)} = 1$, the weighted arithmetic mean operator **WAM**, given by

$$WAM_n(\mathbf{x}) := \sum_{i=1}^{n} w_i^{(n)} x_i$$

is a particular case of positive, single-attribute AGOP

Double-attribute AGOP

Fix an arbitrary sequence of n-tuples of pairs of attributes:

$$n \to (\mathbf{d}^{(n)}, \mathbf{q}^{(n)}) := \{(d_1^{(n)}, q_1^{(n)}), (d_2^{(n)}, q_2^{(n)}), ..., (d_n^{(n)}, q_n^{(n)})\}$$

A sequence $\boldsymbol{H} = \{H_n\}_{n \in \mathbb{N}}$ of double-attribute AFs is a D-AGOP if the following border conditions hold:

$$\liminf_{n \to \infty} H_n((0, d_1^{(n)}, q_1^{(n)}), ..., (0, d_n^{(n)}, q_n^{(n)})) = 0$$

and

$$\limsup_{n \to \infty} H_n((1, d_1^{(n)}, q_1^{(n)}), ..., (1, d_n^{(n)}, q_n^{(n)})) = 1.$$

Monotonic influence of the attributes

- We emphasize the case when all the AFs of a multi-attribute AGOP monotonically influence the results in the same direction.
- We say that a S-AGOP (or a D-AGOP) $\boldsymbol{H}$ is *positive* (*negative*) when every H_n is positive (negative).
- Obviously, the **WAM** is a positive S-AGOP

Monotonic influence of the attributes ••

Consider the generalized root-mean power $\boldsymbol{H}$, given by

$$H_n((x_1, d_1), \ldots, (x_n, d_n)) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{s(n)}\right)^{\frac{1}{s(n)}},$$

where $d_k > 0$ for $k = 1, \ldots, n$. We can show that $\boldsymbol{H}$ is

- negative, if $s(n) = -\sum_{i=1}^{n} d_i$;
- positive, if $s(n) = \sum_{i=1}^{n} d_i$.

Properties for M-AGOPs

The formulation of any property of an AGOP for a M-AGOP requires particular care:

- static properties: AGOPS $\rightarrow$ M-AGOPS;
- q-static properties: AGOPS $\rightarrow$ M-AGOPS;
- dynamic properties: AGOPS $\nrightarrow$ M-AGOPS;

Static properties for a S-AGOP

Consider the following variation $\mathbf{H}_q$ of the generalized root-mean power, with $q \in \mathbb{N}$, given by

$$H_n((x_1,d_1),...,(x_n,d_n)) = \begin{cases} \left(\frac{1}{q}\sum_{i=1}^{n} x_i^{s(n)}\right)^{\frac{1}{s(n)}}, & \text{if } n \le q; \\ \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{s(n)}\right)^{\frac{1}{s(n)}}, & \text{otherwise,} \end{cases}$$

where $\mathbb{J} = \,]0,\infty[$ and $s(n) = \sum_{i=1}^{n} d_i$.

Static properties for a S-AGOP

Obviously $\mathbf{H}_q$ is a S-AGOP and we can show that it is:

- continuous $\longrightarrow$ every AF is continuous;
- symmetric $\longrightarrow$ every AF is symmetric;
- q-idempotent $\longrightarrow$ every AF with arity grater than q is idempotent.

Dynamic properties for a S-AGOP

- In case of a dynamic property, the passage from an AGOP to a S-AGOP is not to be taken for granted.
- The problem of how defining a dynamic property for a M-AGOP has to be considered case by case.

Dynamic neutral element

In case of the concept of dynamic neutral element, a convincing version, where, for sake of simplicity, we limit ourselves to the passage from arity $n = 3$ to $n = 2$, is the following.

- The element $e \in \mathbb{I}$ is a dynamic neutral element for a S-AGOP **H** when

$$H_3((x_1, d_1), (x_2, d_2), (e, d_3)) = H_2((x_1, d_1), (x_2, d_2))$$

for all $x_1, x_2 \in \mathbb{I}$ and for all $d_1, d_2, d_3 \in \mathbb{J}$.

Self-identity for S-AGOPs

The same does not occur for self-identity. If, for instance, we require the equality

$$H_1(x_1,d_1) = H_2\Big((x_1,d_1),(H_1(x_1,d_1),d_2)\Big)$$

for all $x_1, x_2 \in \mathbb{I}$ and for all $d_1, d_2 \in \mathbb{J}$, the arbitrariness in the choice for d_2 suffers the big drawback of imposing extreme rigidity to the property, up to a substantial loss of its significance.

Asymptotic idempotency for S-AGOPs

A S-AGOP **H** is *asymptotically idempotent* if there exists at least a sequence of n-tuples of attributes:

$$n \to \mathbf{d}^{(n)} := \{d_1^{(n)}, d_2^{(n)}, ..., d_n^{(n)}\}$$

such that

$$\lim_{n\to\infty} H_n((x, d_1^{(n)}), ..., (x, d_n^{(n)})) = x \quad \text{for all } x \in \mathbb{I}.$$

An asymptotically idempotent S-AGOP • ○ ○

Let **H** be the S-AGOP given by

$$H_n((x_1, d_1), ..., (x_n, d_n)) = \begin{cases} \max\limits_{i=1,...,n} \{{x_i}^{s(n)}\} \cdot \left(1 - \frac{1}{2n}\right) + \frac{1}{2n}, & n \text{ even;} \\ \max\limits_{i=1,...,n} \{{x_i}^{s(n)}\} \cdot \left(1 - \frac{1}{2n}\right), & n \text{ odd,} \end{cases}$$

where $\mathbb{J} = \mathbb{I}$ and $s(n) := \sum_{i=1}^{n} d_i$.

An asymptotically idempotent S-AGOP • • ◦

We can prove that this operator is asymptotically idempotent. Indeed, for any sequence $(d_1^{(n)}, \ldots, d_n^{(n)})$ such that $s(n) = 1$ we have

$$\lim_{n\to\infty} H_n((x, d_1^{(n)}), \ldots, (x, d_n^{(n)})) = x \quad \text{for all } x \in \mathbb{I}.$$

However, not all the sequences of attributes are good. Indeed, in the case $(d_1^{(n)}, \ldots, d_n^{(n)}) = (1, \ldots, 1)$, we get

$$\lim_{n\to\infty} H_n((x, 1), \ldots, (x, 1)) = 0 \quad \text{for all } x \in \mathbb{I}.$$

An asymptotically idempotent S-AGOP • • •

Let **H** be the S-AGOP given by

$$H_n((x_1, d_1), ..., (x_n, d_n)) = \max(x_1, ..., x_n) \cdot \frac{\sum_{i=1}^{n} {x_i}^{d_i}}{1 + \sum_{i=1}^{n} {x_i}^{d_i}},$$

where $\mathbb{J} = \mathbb{I}$.

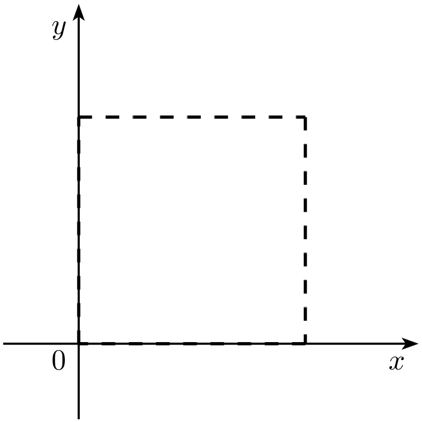
We have the relevant property that such operator is asymptotically idempotent for **any arbitrary** sequence of n-tuples of attributes.

Unanimity as generator of aggregation

- Suppose we have to build an aggregation function but we have no idea how to do.
- Assume that the only principle we are based on is *unanimity*, in the sense that when all the input values are identical, the result must be the common value.
- Now the problem is: what happens when there is not unanimity, i.e. all the data are not equal?

An example

- For sake of simplicity, let us deal with the binary case. We have to build an AF $G_2 : \mathbb{I}^2 \longrightarrow \mathbb{I}$ and we only know its behaviour on the bisector, i.e. $G_2(x,x) = id_{\mathbb{I}}$. How to define $G_2(x_1, x_2)$?
- The principle is: which is the nearest "unanimous point" to (x_1, x_2)? Suppose that the distance is the Euclidean one and let's start with $(0,1)$.
- Its nearest "unanimous point" is $(1/2, 1/2)$. Then, $G_2(0,1) := G_2(1/2, 1/2) = 1/2$.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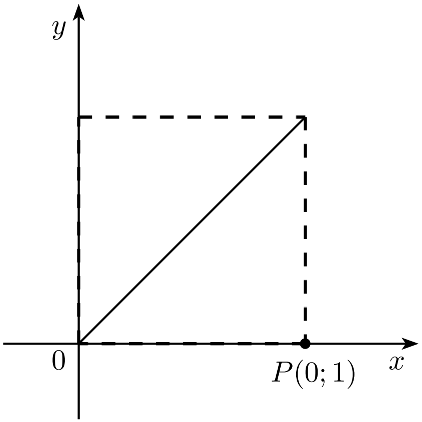
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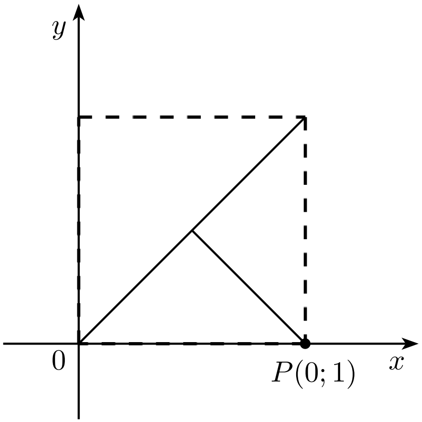
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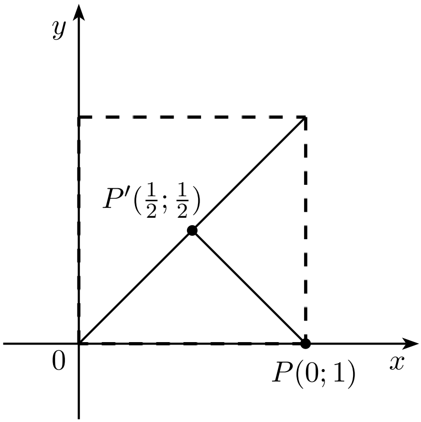
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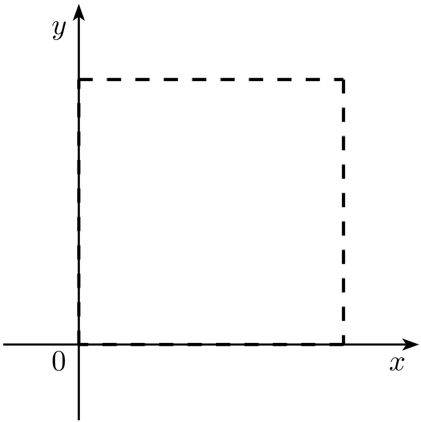
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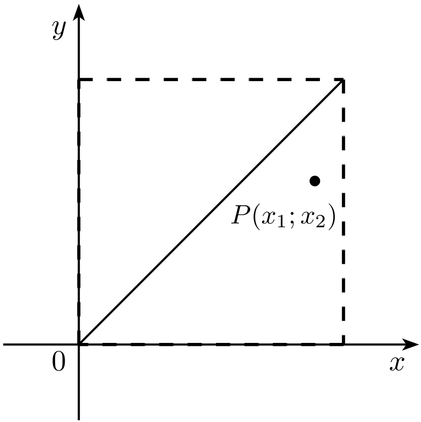
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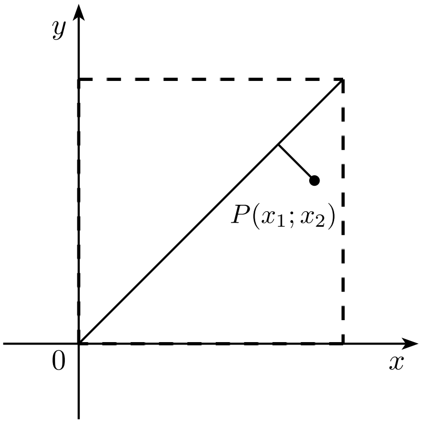
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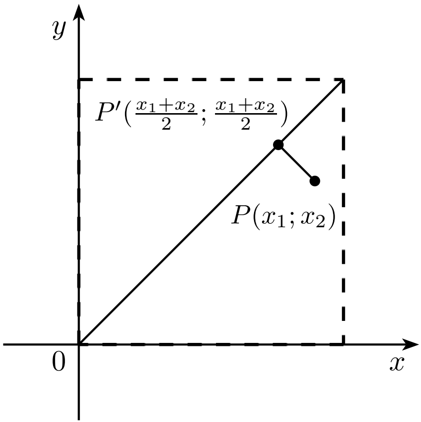
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Weak unanimity

- More generally, given a mapping $f_n : \mathbb{I} \rightarrow \mathbb{I}$, define

$$G_n(\mathbf{x}) = f_n(\arg\min_y \sqrt{(x_1 - y)^2 + \ldots + (x_n - y)^2}).$$

- Unanimity forces $f_n(x) = id_{\mathbb{I}}$.
- Weak unanimity means that $f_n(x) \rightarrow x$ as $n \rightarrow \infty$.

Motivations for weak unanimity

- In many applications weak unanimity is more useful than simple unanimity, because, especially in presence of a huge number of data, it makes the aggregation sensitive to the number of inputs.
- This means that, for instance, weak unanimity can discriminate between a large block of input values and a few positive ones, even if they all are equal or very close each other.
- In mathematical terms, weak and usual unanimity translate into asymptotic and classical idempotency of the AGOP.

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Thanks for your attention