Shall we find normal forms in orthomodular lattices?

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Definitions

ortholattice, modular lattice, modular ortho-
and orthomodular lattice

• an ortholattice is a lattice with an ortho-
complementation

  order-reversing \( a \leq b \Rightarrow b' \leq a' \)
  involution law \( a''' = a \)
  complement law \( a' \lor a = 1 \)
  \( a' \land a = 0 \)
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                        & : a' \land a = 0
  \end{align*}

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\[ a \lor (b \land (a \lor c)) = (a \lor b) \land (a \lor c) \]
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Definitions

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Definitions

word problem, decidability, normal form

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- If there exists such an algorithm, then we say the word problem is **decidable** (solvable), otherwise it is **undecidable** (unsolvable).
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• A **word problem** is the problem of deciding, whether or not two given words represent the same element of the algebra.

• If there exists such an algorithm, then we say the word problem is **decidable (solvable)**, otherwise it is **undecidable (unsolvable)**.

• A **normal form** also called **canonical form** of an object is a standard way of presenting that object.
Definitions

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- A **word problem** is the problem of deciding, whether or not two given words represent the same element of the algebra.

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- A **normal form** also called **canonical form** of an object is a standard way of presenting that object. Philip M. Whitman proved that for all equal elements in a free lattice, there is one of shortest length.
Max W. Dehn knew that the word problem was difficult, he wrote:

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Skolem (1920) solved the (uniform) word problem for finitely presented lattices.
What about quantum logic?

Is the word problem related to quantum logic decidable?
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First results:
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- the word problem remains an open challenge in the **orthomodular** as well as in the **modular-ortho** case (Herrmann, Micol and Roddy)
- for the **free orthomodular lattices over two generators**, the problem is decidable
Motivation of today’s talk

The absence of distributivity in OMLs makes it difficult to find the normal forms of complex expressions. Some tools were developed to overcome this problem:
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Computation in \( F(a, b) \) (Navara)
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The absence of distributivity in OMLs makes it difficult to find the normal forms of complex expressions. Some tools were developed to overcome this problem:

- Foulis-Holland Theorem
- Focusing technique (Greechie)
- Computation in $F(a, b)$ (Navara)

In all three techniques the commuting elements play a crucial role

$$xCy \text{ if } x = (x \land y) \lor (x \land y')$$
For which of the 96 binary operations \( \ast \) in an OML, the following implication holds:

\[
x \leq y \quad \Rightarrow \quad x \ast z \leq y \ast z
\]

where \( x, y \) and \( z \) are elements of an OML.
Why this question?

Is it possible to find operations $\ast$ for which

$$(a_1 \ast b) \land (a_2 \ast b) \land \ldots \land (a_n \ast b) = (a_1 \land \ldots \land a_n) \ast b$$

holds?
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Tools

- There are $2^4 = 16$ binary Boolean operations, each of them represents 6 OML operations (Megill, Pavičić)
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Tools

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• Navara’s computation in $F(a, b, c)$

• Kalmbach embedding

• Computer program (Hyčko)
  http://www.mat.savba.sk/~hycko/oml
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Boolean algebra

For which of the 16 binary operations $*$ in an Boolean algebra, the following implication holds:

$$x \leq y \implies x * z \leq y * z$$

for $x, y$ and $z$, elements of an Boolean algebra.
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Boolean algebra

From the Boolean operations only two do not fulfil the equation

\[ x \leq y \implies x \ast z \leq y \ast z \]
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**Boolean algebra**

From the Boolean operations only two do not fulfil the equation

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the equivalence \( \leftrightarrow \)

\[ a \leftrightarrow b = (a \land b) \lor (a' \land b') = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise}, \end{cases} \]
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Boolean algebra

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and its complement \( \not\leftrightarrow \)

\[ a \not\leftrightarrow b = (a \land b') \lor (a' \land b) = \begin{cases} 0 & \text{if } a = b, \\ 1 & \text{otherwise}. \end{cases} \]
Corresponding in an OML

These two operations correspond to 12 OML operations
Corresponding in an OML

These two operations correspond to 12 OML operations

- For the equivalence $\leftrightarrow$

$\begin{align*}
(x \land y) \lor (x' \land y') & \quad \text{Beran 8} \\
(x' \lor y) \land [x \lor (x' \land y')] & \quad \text{Beran 24} \\
(x \lor y') \land [y \lor (x' \land y')] & \quad \text{Beran 40} \\
(x' \lor y) \land [y' \lor (x \land y)] & \quad \text{Beran 56} \\
(x \lor y') \land [x' \lor (x \land y)] & \quad \text{Beran 72} \\
(x' \lor y) \land (x \lor y') & \quad \text{Beran 88}
\end{align*}$
Corresponding in an OML

- For its complement $\leftrightarrow$

$$\begin{align*}
(x \land y') \lor (x' \land y) & \quad \text{Beran 9} \\
(x' \lor y') \land [x \lor (x' \land y)] & \quad \text{Beran 25} \\
(x' \lor y') \land [y \lor (x \land y')] & \quad \text{Beran 41} \\
(x \lor y) \land [y' \lor (x' \land y)] & \quad \text{Beran 57} \\
(x \lor y) \land [x' \lor (x \land y')] & \quad \text{Beran 73} \\
(x \lor y) \land (x' \lor y') & \quad \text{Beran 89}
\end{align*}$$
Calculation in $F(a, b, c)$

For two elements $x, y$ of an OML, it is said $x$ commutes with $y$ and written $xCy$ if

$$x = (x \land y) \lor (x \land y'),$$

where

$cCa$ and $cCb$. 

M. Navara described the OML $F(a, b, c)$ generated by three generators $a, b, c$.
Calculation in $F(a, b, c)$

For two elements $x, y$ of an OML, it is said $x$ commutes with $y$ and written $xCy$ if

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It is easy to show that the following holds:

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M. Navara described the OML $F(a, b, c)$ generated by three generators $a, b, c$ where $cCa$ and $cCb$. 
Calculation in $F(a, b, c)$

The generators $a$, $b$ and $c$ can be expressed by Navara’s notation:

\[
\begin{align*}
    a &= a \left( \begin{array}{c}
    \bigcirc \\
    \bigcirc
    \end{array}, \begin{array}{c}
    \bigcirc \\
    \bigcirc
    \end{array} \right)_c b \\
    b &= a \left( \begin{array}{c}
    \bigcirc \\
    \bigcirc
    \end{array}, \begin{array}{c}
    \bigcirc \\
    \bigcirc
    \end{array} \right)_c b \\
    c &= a \left( \begin{array}{c}
    \bigcirc \\
    \bigcirc
    \end{array}, \begin{array}{c}
    \bigcirc \\
    \bigcirc
    \end{array} \right)_c b.
\end{align*}
\]
Calculation in $F(a, b, c)$

The generators $a$, $b$ and $c$ can be expressed by Navara’s notation:

\[ a = a \left( \begin{array}{c|c} \circ & \circ \\ \hline \circ & \circ \end{array} \right)_c b \]

\[ b = a \left( \begin{array}{c|c} \circ & \circ \\ \hline \circ & \circ \end{array} \right)_c b \]

\[ c = a \left( \begin{array}{c|c} \circ & \circ \\ \hline \circ & \circ \end{array} \right)_c b. \]

\[ a \land c \in [0, c] \]

\[ b \land c' \in [0, c'] \]
Calculation in $F(a, b, c)$

Our initial question:
Calculation in $F(a, b, c)$

Our initial question:
For which binary operations $\ast$ holds:

$$x \leq y \Rightarrow x \ast z \leq y \ast z$$

where $x$, $y$ and $z$ are elements of an OML,
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Calculation in $F(a, b, c)$

Our initial question:
For which binary operations $\ast$ holds:

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Calculation in $F(a, b, c)$

Our initial question:
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Our initial question:
For which binary operations \( * \) holds:

\[
x \leq y \implies x * z \leq y * z
\]

where \( x, y \) and \( z \) are elements of an OML,
becomes by substituting \( x \) by \( (a \land c) \), \( y \) by \( c \) and \( z \) by \( b \).
Our initial question: 
For which binary operations $*$ holds:

$$x \leq y \implies x \ast z \leq y \ast z$$

where $x$, $y$ and $z$ are elements of an OML, becomes by substituting $x$ by $(a \land c)$, $y$ by $c$ and $z$ by $b$. 
For which binary operations $*$ holds:

$$(a \land c) \ast b \leq c \ast b$$

where $a$, $b$ and $c$ are the generators of $F(a, b, c)$. 
**Calculation in** $F(a, b, c)$

• By choosing $c = 1$: 

By choosing $c = 1$: 

We could eliminate further 40 operations from our list of candidates.

Do the remaining 44 binary operations fulfil our equation?
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Calculation in $F(a, b, c)$

- By choosing $c = 1$:

  $$a \ast b \leq 1 \ast b$$
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Kalmbach embedding

- A method of embedding any arbitrary poset $P$ into a concrete OML $L = K(L)$, used by Kalmbach and extended by Harding (1991) and Mayet and Navara (1995).
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- Given any poset $P$, then there exists an OML $L$ and an embedding
  $\phi : P \rightarrow L$
Kalmbach embedding

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- Given any poset $P$, then there exists an OML $L$ and an embedding
  \[ \phi : P \rightarrow L \]
  for $x, y \in P$

  1. $x \leq y \iff \phi(x) \leq \phi(y)$
  2. if $x \land y$ exists, then $\phi(x) \land \phi(y) = \phi(x \land y)$
  3. if $x \lor y$ exists, then $\phi(x) \lor \phi(y) = \phi(x \lor y)$
• A method of embedding any arbitrary poset $P$ into a concrete OML $L = K(L)$, used by Kalmbach and extended by Harding (1991) and Mayet and Navara (1995).

• Given any poset $P$, then there exists an OML $L$ and an embedding
  $$\phi : P \rightarrow L$$
for $x, y \in P$

  1. $x \leq y \iff \phi(x) \leq \phi(y)$
  2. if $x \wedge y$ exists, then $\phi(x) \wedge \phi(y) = \phi(x \wedge y)$
  3. if $x \vee y$ exists, then $\phi(x) \vee \phi(y) = \phi(x \vee y)$

• $L$ can then be embedded into a Boolean algebra by preserving the lattice operations.
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Kalmbach embedding

Figure: Greechie- and Hasse diagram of $2^3 \oplus 2^2$
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Kalmbach embedding

Figure: Greechie- and Hasse diagram of $2^3 \oplus 2^2$

Which is the Kalmbach embedding of the pentagon.
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Kalmbach embedding
Monotony in the first argument

This demonstrates that, for exploring the monotonicity in the first argument, we can discard all the operations with Beran’s number in \( \{65, 66, 67, \ldots, 80\} \).
Kalmbach embedding
Monotony in the first argument

This demonstrates that, for exploring the monotonicity in the first argument, we can discard all the operations \( \ast \) with Beran’s number in \( \{65, \ldots, 80\} \).
Shall we find normal forms in orthomodular lattices?

Kalmbach embedding
Monotony in the first argument
Kalmbach embedding
Monotony in the first argument

This demonstrates that, for exploring the monotonicity in the first argument, we can discard all the operations $*$ with Beran’s number in $\{65, \ldots, 80\}$. 
This demonstrates that, for exploring the monotonicity in the first argument, we can discard all the operations $\ast$ with Beran’s number in $\{65, \ldots, 80\}$. Further, we can even discard all the binary operations containing $a'$. 
Results

Monotony in the first argument

At the end only 17 binary operations come into question to be non-decreasing in the first argument, they are the operations with Beran’s number and Navara’s notation:
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Results

Monotony in the first argument

At the end only 17 binary operations come into question to be non-decreasing in the first argument, they are the operations with Beran’s number and Navara’s notation:

1 \( \clubsuit \odot \), 2 \( \clubsuit \odot \), 3 \( \bullet \odot \), 6 \( \bullet \odot \),
At the end only 17 binary operations come into question to be non-decreasing in the first argument, they are the operations with Beran’s number and Navara’s notation:

\[ \begin{align*}
1 & \quad 2 \quad 3 \quad 6 \quad 22 \quad 34 \quad 38 \quad 44 \quad 51 \quad 54 \quad 58 \quad 61 \quad 86 \quad 92 \quad 93 \quad 96
\end{align*} \]
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Results

Monotony in the first argument

At the end only 17 binary operations come into question to be non-decreasing in the first argument, they are the operations with Beran’s number and Navara’s notation:

1, 2, 3, 6, 22, 34, 38, 39, 44, 86, 92, 93, 96.
Results

Monotony in the first argument

At the end only 17 binary operations come into question to be non-decreasing in the first argument, they are the operations with Beran’s number and Navara’s notation:

1 \[\text{\textbullet\textbullet\textbullet\textbullet}\], 2 \[\text{\textbullet\textbullet\textbullet\textbullet}\], 3 \[\text{\textbullet\textbullet\textbullet\textbullet}\], 6 \[\text{\textbullet\textbullet\textbullet\textbullet}\],

22, \[\text{\textbullet\textbullet\textbullet\textbullet}\]

34 \[\text{\textbullet\textbullet\textbullet\textbullet}\], 38 \[\text{\textbullet\textbullet\textbullet\textbullet}\], 39 \[\text{\textbullet\textbullet\textbullet\textbullet}\], 44 \[\text{\textbullet\textbullet\textbullet\textbullet}\],

51 \[\text{\textbullet\textbullet\textbullet\textbullet}\], 54 \[\text{\textbullet\textbullet\textbullet\textbullet}\], 58 \[\text{\textbullet\textbullet\textbullet\textbullet}\], 61 \[\text{\textbullet\textbullet\textbullet\textbullet}\].
Results

Monotony in the first argument

At the end only 17 binary operations come into question to be non-decreasing in the first argument, they are the operations with Beran’s number and Navara’s notation:

1 $\circ\circ \cdot \circ\circ$, 2 $\circ\circ \cdot \circ\bullet$, 3 $\circ\bullet \cdot \circ\circ$, 6 $\circ\bullet \cdot \circ\bullet$,
22, $\circ\circ\circ\circ$
34 $\circ\bullet\bullet\circ$, 38 $\bullet\circ\bullet\circ$, 39 $\circ\bullet\bullet\bullet$, 44 $\circ\bullet\bullet\circ$,
51 $\circ\bullet\bullet\circ\circ$, 54 $\bullet\circ\bullet\bullet$, 58 $\bullet\bullet\bullet\circ$, 61 $\bullet\bullet\bullet\bullet$
86 $\bullet\bullet\circ\circ$, 92 $\bullet\bullet\circ\bullet$, 93 $\bullet\bullet\bullet\circ$ and 96 $\bullet\bullet\bullet\bullet$. 
Results

These are exactly the binary operations for which holds

\[ x \leq y \implies x \ast z \leq y \ast z \]
These are exactly the binary operations for which holds
\[ x \leq y \implies x \ast z \leq y \ast z \]

We found also 17 operations which are non-decreasing in the second argument,
Shall we find normal forms in orthomodular lattices?

Results

6 operations are non-decreasing in both arguments,
Shall we find normal forms in orthomodular lattices?

Results

6 operations are non-decreasing in both arguments,

0  Beran's number 1  """"

""""  """"  """"  ••••
6 operations are non-decreasing in both arguments,

0  Beran's number 1

\( a \land b \)  Beran's number 2
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Results

6 operations are non-decreasing in both arguments,

0  Beran’s number 1

\( a \land b \)  Beran’s number 2

\( a \)  Beran’s number 22
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Results

6 operations are non-decreasing in both arguments,

0       Beran's number 1

\[ a \wedge b \]
Beran's number 2

\[ a \]
Beran's number 22

\[ b \]
Beran's number 39
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**Results**

6 operations are non-decreasing in both arguments,

- 0  Beran’s number 1
- $a \land b$  Beran’s number 2
- $a$  Beran’s number 22
- $b$  Beran’s number 39
- $a \lor b$  Beran’s number 92
Results

6 operations are non-decreasing in both arguments,

0  Beran’s number 1  

\(a \land b\)  Beran’s number 2  

\(a\)  Beran’s number 22  

\(b\)  Beran’s number 39  

\(a \lor b\)  Beran’s number 92  

1  Beran’s number 96
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Results
We also reversed our initial inequation For which of the 96 binary operations $*$ in an OML, the following inequality holds:

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Results

We also reversed our initial inequation For which of the 96 binary operations $\ast$ in an OML, the following inequality holds:

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0 Beran’s number 1
Results

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$$x \leq y \implies x \ast z \geq y \ast z$$

0 Beran’s number 1

$a' \land b'$ Beran’s number 5
Results

We also reversed our initial inequation. For which of the 96 binary operations $\ast$ in an OML, the following inequality holds:

$$x \leq y \Rightarrow x \ast z \geq y \ast z$$

- $0$ Beran’s number 1
- $a' \land b'$ Beran’s number 5
- $a'$ Beran’s number 75
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Results

We also reversed our initial inequation For which of the 96 binary operations $*$ in an OML, the following inequality holds:

$$x \leq y \Rightarrow x \ast z \geq y \ast z$$

- 0 Beran’s number 1
- $a' \land b'$ Beran’s number 5
- $a'$ Beran’s number 75
- $b'$ Beran’s number 58
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\end{align*}
\]
Shall we find normal forms in orthomodular lattices?

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After all there are 46 binary operations which fulfil

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in the first or second argument
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and if \( a \ast b \) is one of them then also

\[ a' \ast b \]

\[ b \ast a \]

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Conclusions

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Conclusions

Our aim was to reduce the complexity of some OML operations and to find a way to write them in a unique normal form. Therefore we need to find two operations which satisfy the distributive law. One of these operations need also to be associative and commutative. The only two operations which fulfil the three conditions are the join and the meet. so, we had to find the operations which distribute over the meet

\[(a_1 \ast b) \land (a_2 \ast b) \land \ldots \land (a_n \ast b) = (a_1 \land \ldots \land a_n) \ast b\]
Shall we find normal forms in orthomodular lattices?

Conclusions

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Shall we find normal forms in orthomodular lattices?

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- Thank you for your attention!
- Questions?
Shall we find normal forms in orthomodular lattices?

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Shall we find normal forms in orthomodular lattices?

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Definitions

- an **ortholattice** is a lattice with an ortho-complementation
  
  order-reversing \( a \leq b \Rightarrow b' \leq a' \)

  involution law \( a'' = a \)

  complement law \( a' \lor a = 1 \)

  \( a' \land a = 0 \)

  
  

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