Domains of fuzzy probability II.

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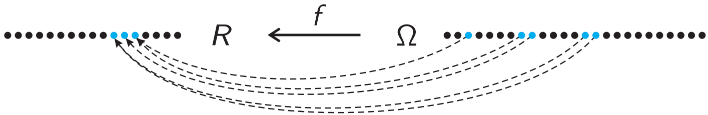
RANDOM VARIABLE (classics)

$$\cdots\cdots\cdots \quad R \xleftarrow{\ f\ } \Omega \quad \cdots\cdots\cdots$$

- $(\Omega, \mathbb{A}, P)$, $(R, \mathbb{B}_R, P_f)$, $\quad f: \ \Omega \to R \quad$ ... measurable map
- $\mathbb{A}$ ... σ-algebra of events, $\quad \mathbb{B}$ ... Borel measurable sets
- $\forall B \in \mathbb{B}_R \ \exists A \in \mathbb{A}: \quad f^{\leftarrow}(B) = A$

RANDOM VARIABLE (classics)



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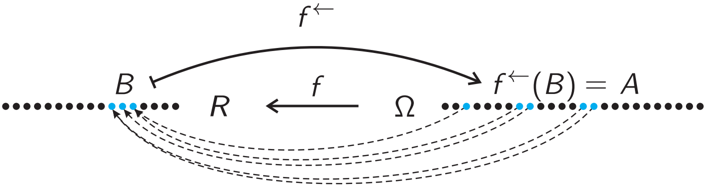
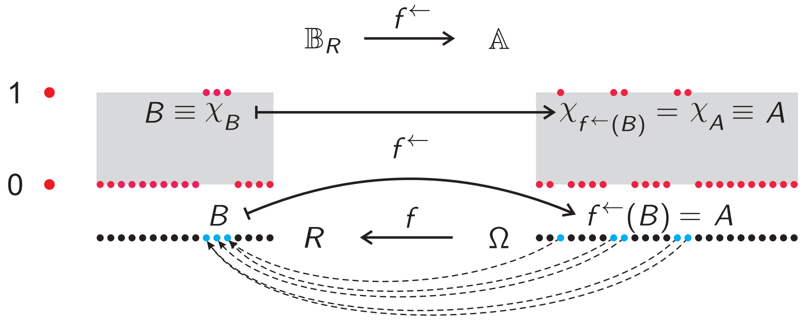
RANDOM VARIABLE (classics)

$$\mathbb{B}_R \xrightarrow{\ f^{\leftarrow}\ } \mathbb{A}$$



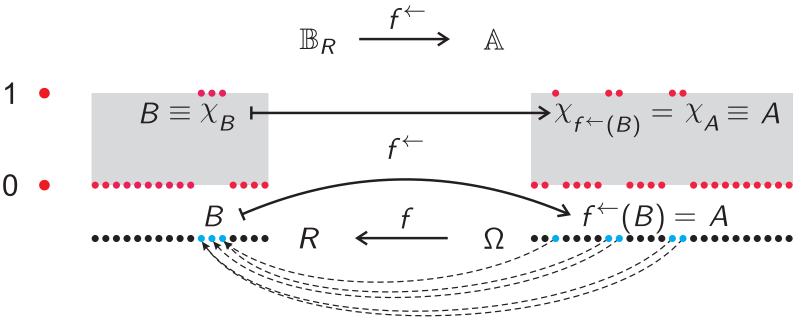
- Borel measurable set $B \in \mathbb{B}_R$, event $f^{\leftarrow}(B) = A \in \mathbb{A}$
- $B \longmapsto f^{\leftarrow}(B) = A$
- $f^{\leftarrow} : \mathbb{B}_R \to \mathbb{A}$

RANDOM VARIABLE (classics)



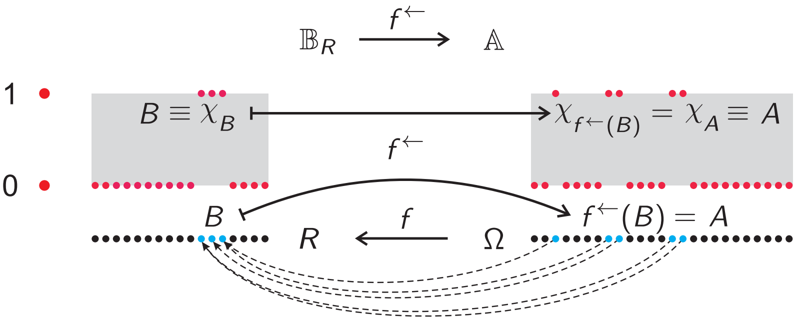
- $B \equiv$ its indicator function χ_B
- $f^{\leftarrow}(B) \equiv \chi_{f^{\leftarrow}(B)} = \chi_A$
- $\forall \omega \in \Omega : \quad \chi_B(f(\omega)) = \chi_A(\omega)$

RANDOM VARIABLE (classics)



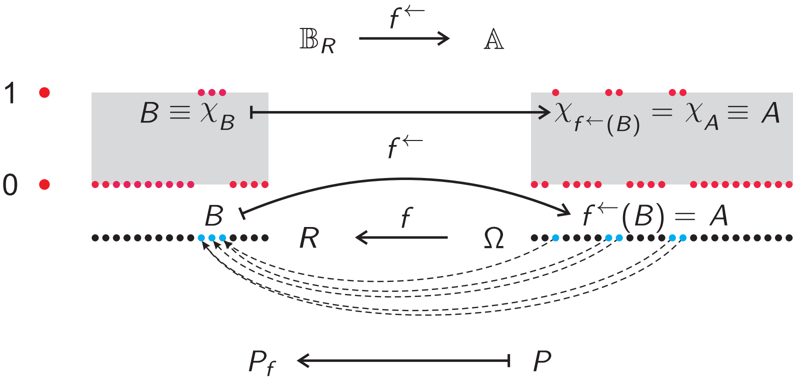
- $\forall \omega \in \Omega : \quad \chi_B(f(\omega)) = \chi_A(\omega)$
- f is measurable **iff** for every $B \in \mathbb{B}_R$ the composition $\chi_B \circ f$ is the indicator function of some $A \in \mathbb{A}$

RANDOM VARIABLE (classics)



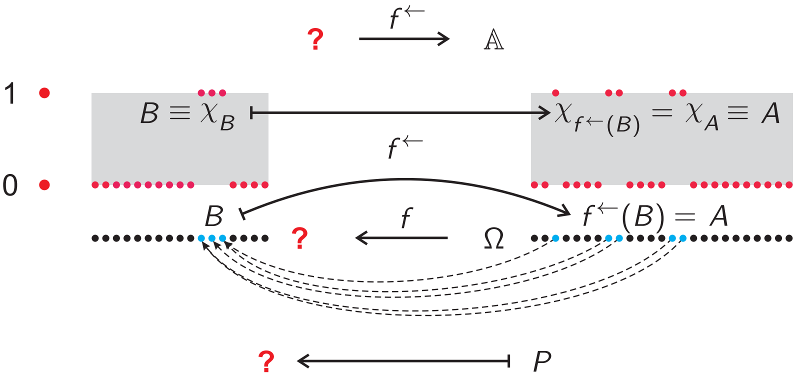
- $f: \ \Omega \to R$ . . . measurable map
- $f^{\leftarrow}: \ \mathbb{B}_R \to \mathbb{A}$ . . . Boolean homomorphism – $f^{\leftarrow}(R) = \Omega$, $f^{\leftarrow}(\{\}) = \{\}$, and preserves $\wedge$, $\vee$, complement
- $f^{\leftarrow}$ . . . classical observable

RANDOM VARIABLE (classics)



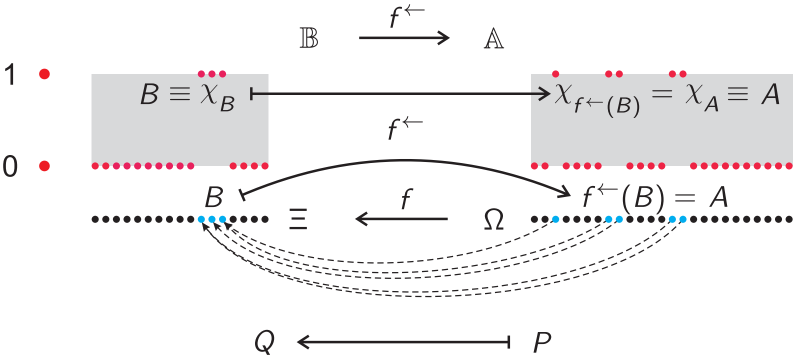
- $P_f(B) = P(A)$
- f ...sends P to P_f, the distribution of f
- in fact f yields a transformation $T_f : \mathcal{P}(\mathbb{A}) \to \mathcal{P}(\mathbb{B}_R)$

RANDOM TRANSFORMATION (classics)



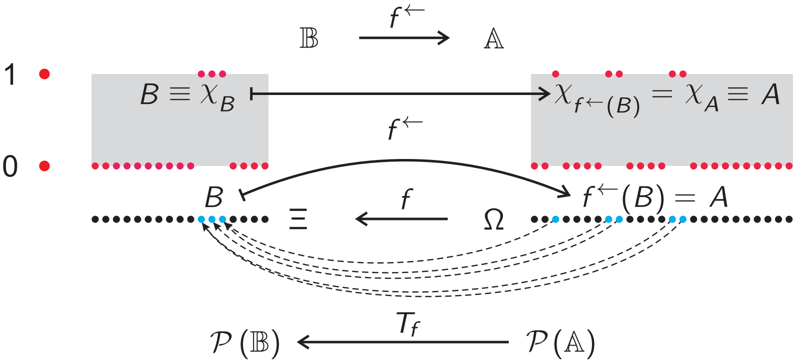
- $?(B) = P(A)$
- f ...sends P to ?, the distribution of f
- $f: \Omega \to ?$

TRANSFORMATION (still classics)



- $(\Omega, \mathbb{A}, P)$, $(\Xi, \mathbb{B}, Q)$, $\quad f : \Omega \to \Xi$ ... measurable map
- $\mathbb{A}$, $\mathbb{B}$ ... σ-algebras of events
- $f^{\leftarrow} : \mathbb{B} \to \mathbb{A}$ ... Boolean homomorphism; it is called OBSERVABLE

TRANSFORMATION (still classics)



- $T_f : \mathcal{P}(\mathbb{A}) \to \equiv \mathcal{P}(\mathbb{B})$ ...transformation (statistical map)
- $T_f \restriction \Omega = f$
- OBSERVATION: There is a duality between (classical) transformations and observables.

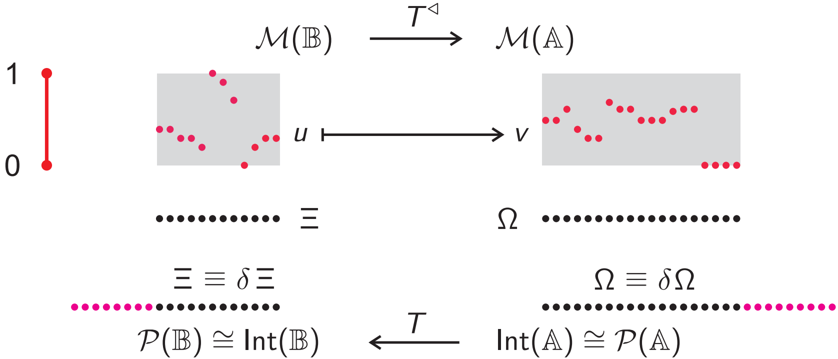
FUZZIFICATION (conservative)

L.A. Zadeh proposed the following fuzzification of probability:

- to extend $\mathbb{A}$ (classical events) to $\mathcal{M}(\mathbb{A})$ (fuzzy events);
- instead of $P \in \mathcal{P}(\mathbb{A})$ to use $\int(\cdot)\mathrm{d}P$ (fuzzy probability measure).

Denote $\mathrm{Int}(\mathbb{A}) = \{\int(\cdot)\mathrm{d}P;\ P \in \mathcal{P}(\mathbb{A})\}$. Observe that $\mathbb{A} \mapsto \mathcal{M}(\mathbb{A}) \mapsto \mathbb{A}$, resp. $P \mapsto \int(\cdot)\mathrm{d}P \mapsto P$, yields a one-to-one correspondence between classical random fields and fuzzy random fields, resp. classical probabilities and fuzzy probabilities. This leads to the following fuzzifications of our transformation scheme (previous slide).

FUZZY RANDOM VARIABLE (already non-classics)



PROBLEM. Define a suitable fuzzy transformation $T: \mathrm{Int}(\mathbb{A}) \to \mathrm{Int}(\mathbb{B})$ and its dual fuzzy observable $T^{\triangleleft}: \mathcal{M}(\mathbb{A}) \to \mathcal{M}(\mathbb{B})$ such that it extends the classical duality between $T_f: \mathcal{P}(\mathbb{A}) \to \mathcal{P}(\mathbb{B})$ and $f^{\leftarrow}: \mathbb{B} \to \mathbb{A}$.

SOLUTION

IDEA: To get information about $\mathcal{M}(\mathbb{B})$, via $T^{\triangleleft}$, using the available information about $\mathcal{M}(\mathbb{A})$:
a map $f : \ \Omega \to \Xi$ is **fuzzy measurable** if for each $u \in \mathcal{M}(\mathbb{B})$ the composition $u \circ f$ belongs to $\mathcal{M}(\mathbb{A})$ and the induced dual map $T^{\triangleleft} : \ \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{A})$ $(T^{\triangleleft}(u) = u \circ f)$ "preserves the structure of fuzzy random events".

SOLUTION

In general: Consider $T : \text{Int}(\mathbb{A}) \to \text{Int}(\mathbb{B})$.
In the classical case T maps each degenerated integral (with respect to a degenerated point measure) into a degenerated integral. To model some quantum phenomena we have to assume that in general T maps a degenerated integral on $\mathcal{M}(\mathbb{A})$ into a genuine non-degenerated integral $\int(\cdot)\mathrm{d}Q$, where Q is a genuine probability measure on $\mathbb{B}$.

SOLUTION: A map $T : \text{Int}(\mathbb{A}) \to \text{Int}(\mathbb{B})$ is **fuzzy measurable** (fuzzy transformation) if the "fuzzy composition" $u \diamond T$ defined by

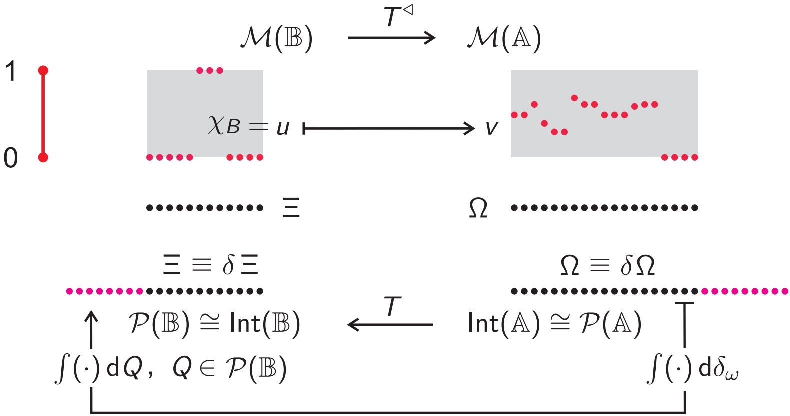
$$(u \diamond T)(\omega) = \int u \,\mathrm{d}Q, \quad \omega \in \Omega, \quad \int(\cdot)\,\mathrm{d}Q = T(\int(\cdot)\,\mathrm{d}\delta_\omega)$$

belongs to $\mathcal{M}(\mathbb{A})$. This defines the dual **fuzzy observable** $T^{\triangleleft} : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{A})$ and $T^{\triangleleft}$ "preserves the structure of fuzzy random events".

OBSERVATIONS

- Each classical observable $f^{\leftarrow} : \mathbb{B} \to \mathbb{A}$ can be uniquely extended to a fuzzy observable $T^{\triangleleft} : \mathcal{M}(\mathbb{B}) \longrightarrow \mathcal{M}(\mathbb{A})$;
- Classical observables = special case of fuzzy observables;
- T can send a degenerated integral to a non-degenerated integral $\Longrightarrow$ $T^{\triangleleft}$ can send a crisp event $u = \chi_B$ to a genuine fuzzy event $v = T^{\triangleleft}(u)$.

GENUINE FUZZY OBSERVABLE

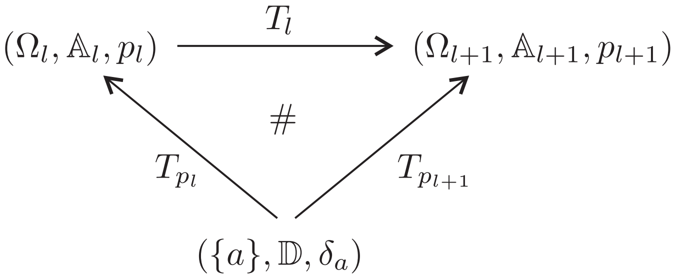


GENERALIZED RANDOM WALK

- In case of finite probability spaces each fuzzy transformations can be consider as generalized random walk.
- Indeed, let $(\Omega, \mathbb{A}, P)$ be finite probability space, let $(\Xi, \mathbb{B})$ be finite measurable space, and let T_Ω be a map of Ω into $\mathcal{P}(\mathbb{B})$.
- Then there exist unique fuzzy transformation T (consider as a map of $\mathcal{P}(\mathbb{A})$ into $\mathcal{P}(\mathbb{B})$) such that $T_\Omega(\omega) = T(\delta_\omega) \in \mathcal{P}(\mathbb{B}),\ \omega \in \Omega$, and $T(\delta_\omega)$ can be considered as the probability of transitions from $\omega \in \Omega$ to points of Ξ.
- More information about generalized random walks can be found in
  FRIČ, R., PAPČO, M.: Statistical maps and generalized random walks. Math. Slovaca. (To appear.)

Definition

For a positive natural number k, let $\{(\Omega_l, \mathbb{A}_l, p_l)\}_{l=1}^{k+1}$ be a sequence of discrete probability spaces and let $\{T_l\}_{l=1}^{k}$ be a sequence of extended random maps of $(\Omega_l, \mathbb{A}_l, p_l)$ to $(\Omega_{l+1}, \mathbb{A}_{l+1}, p_{l+1})$ such that the diagram composed of all constituent commutative triangle diagrams



$l = 1, 2, \ldots, k$, is commutative. Then the resulting composed diagram is said to be a *generalized (finite) random walk*.

Example

Consider the following special case of a generalized random walk: $\Omega_l = \Omega_{l+1}$, $l = 1, 2, \ldots, k$, and there is a stochastic matrix $\mathbf{A} = (a_{ij})_{m \times m}$ such that $T_l = T_{\mathbf{A}}$, $l = 1, 2, \ldots, k$. Then $\mathbf{A}$ can be considered as the matrix of transitional probabilities of a Markov chain (with the initial distribution $\mathbf{p}_1$) and the composed diagram describes "k transitions".