Domains of fuzzy probability I.

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Abstract

Since the pioneering paper by L. A. Zadeh
ZADEH, L. A.: *Probability measures of fuzzy events.* J. Math. Anal. Appl. 23 (1968), 421–427.
fuzzy probability theory underwent a considerable development. In
FRIČ, R. and PAPČO, M.: *On probability domains II.* Internat. J. Theoret. Phys. 50, (2011), 3778–3786
we have proposed a classification scheme of ID-posets related to fuzzy probability theory. The classification leads to the understanding of the transition from classical to fuzzy probability.

Preliminaries

D-posets have been introduced by F. Chovanec and F. Kôpka in order to model events in quantum probability. They generalize Boolean algebras, MV-algebras and other probability domains and provide a category in which observables and states become morphisms. Recall that a D-poset is a partially ordered set with the greatest element 1_X, the least element 0_X, and a partial binary operation called *difference*, such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_X = a$ for each $a \in X$;

(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

Fundamental to applications are D-posets of fuzzy sets, i.e. systems $\mathcal{X} \subseteq I^X$ carrying the coordinatewise partial order, coordinatewise convergence of sequences, containing the top and bottom elements of I^X, and closed with respect to the partial operation difference defined coordinatewise.

D-posets of fuzzy sets

Denote ID the category having D-posets of fuzzy sets as objects and having sequentially continuous D-homomorphisms as morphisms. Objects of ID are subobjects of the powers I^X. Recall that in ID "everything is determined" by the cogenerator $I = [0, 1]$ considered as a D-poset: objects live inside Cartesian powers I^X and carry the initial D-poset structure with respect to morphisms to I (in case of classical random events sequentially continuous D-homomorphisms to I are exactly probability measures the σ-additivity of which is equivalent to sequential continuity – remember the Lebesgue Dominated Convergence Theorem). The classical probability theory and the fuzzy probability theory, CPT and FPT for short, are two distinguished probability theories and we show that within the category ID the relationship between CPT and FPT can be described.

Canonical examples

Let $\mathbf{A}$ be a σ-field of subsets of Ω and let $\mathcal{M}(\mathbf{A})$ be the MV-algebra of all measurable functions ranging in I; it is called a **generated Łukasiewicz tribe**.

- $\mathbf{A}$ can be considered as an ID-poset: identify $A \in \mathbf{A}$ and its indicator (characteristic) function and define $A \ominus B = A \setminus B$ whenever $B \subseteq A$.
- Let $\mathcal{P}(\mathbf{A})$ be the set of all probability measures on $\mathbf{A}$. Then each $p \in \mathcal{P}(\mathbf{A})$ is an ID-morphism of $\mathbf{A}$ into $I = I^{\{a\}}$. Denote $ev(A) = \{p(A);\ p \in \mathcal{P}(\mathbf{A})\}$ and $ev(\mathbf{A}) = \{ev(A);\ A \in \mathbf{A}\}$. For $X = \mathcal{P}(\mathbf{A})$ and $ev(\mathbf{A}) = \mathcal{X} \subseteq I^X$, $\mathcal{X}$ is a typical ID-poset.
- Analogously, $\mathcal{M}(\mathbf{A})$ can be viewed as an ID-poset.

Why ID-posets?

D-poset ... a "minimal" structure

- 1_X ... sure event
- 0_X ... impossible event
- partial order and difference ... "minimal" partial operation
- $u^* = 1_X - u$ ... negation (complement)

ID-poset ... a stuctured system of fuzzy sets

- "events" ... represented as functions (fuzzy sets) and determined by states (generalized probability measures)
- natural partial order and partial difference ... pointwise
- subtractivity and continuity ... sufficient information to reconstruct classical notions (observable, probability)

Properties of ID-posets

classical:

- sublattice of $[0,1]^X$ ... conjunction, disjunction
- closedness ... limits

fuzzy:

- divisibility ... quantum phenomena (degenerated fuzzy random variables)

Next, we summarize some results from
Frič, R. and Papčo, M.: On probability domains II. Internat. J. Theoret. Phys. **50** (2011), 3778–3786.

Classification of ID-posets

Definition

Let $\mathcal{X} \subseteq I^X$ be an *ID*-poset. If $\mathcal{X}$ is sequentially closed in I^X, then it is said to be *closed*.

Define $\vee$ and $\wedge$ coordinatewise:
$(u \vee v)(x) = u(x) \vee v(x)$ and $(u \wedge v)(x) = u(x) \wedge v(x)$.

Definition

Let $\mathcal{X} \subseteq I^X$ be an *ID*-poset. If $u \vee v \in \mathcal{X}$ and $u \wedge v \in \mathcal{X}$ whenever $u, v \in \mathcal{X}$, then $\mathcal{X}$ is said to be a *lattice ID*-poset.

Definition

Let $\mathcal{X} \subseteq I^X$ be an *ID*-poset and let $n \in \mathbb{N}^+$. If for each $u \in \mathcal{X}$ there exists $v \in \mathcal{X}$ such that $u = nv$, then $\mathcal{X}$ is said to be *divisible by n*. If $\mathcal{X}$ is divisible by n for all $n \in \mathbb{N}^+$, then $\mathcal{X}$ is said to be *divisible*.

Characterization I

Theorem

Let $\mathcal{X} \subseteq I^X$ be an ID-poset. Then the following are equivalent:

(i) *$\mathcal{X}$ is a lattice ID-poset;*

(ii) *$\mathcal{X}$ is a bold algebra.*

Corollary

Let $\mathcal{X} \subseteq \{0,1\}^X$ be an ID-poset. Then the following are equivalent:

(i) *$\mathcal{X}$ is a lattice ID-poset;*

(ii) *$\mathcal{X}$ is a field of sets.*

Characterization II

Corollary

Let $\mathcal{X} \subseteq I^X$ be a closed ID-poset. Then the following are equivalent:

(i) *$\mathcal{X}$ is a lattice ID-poset;*

(ii) *$\mathcal{X}$ is a Łukasiewicz tribe.*

Corollary

Let $\mathcal{X} \subseteq \{0, 1\}^X$ be a closed ID-poset. Then the following are equivalent:

(i) *$\mathcal{X}$ is a lattice ID-poset;*

(ii) *$\mathcal{X}$ is a σ-field.*

Corollary

Let $\mathcal{X} \subseteq I^X$ be a closed and divisible ID-poset. Then the following are equivalent:

(i) *$\mathcal{X}$ is a lattice ID-poset;*

(ii) *There exists a σ-field $\mathbf{A}$ of subsets of X such that $\mathcal{X} = \mathcal{M}(\mathbf{A})$.*

Corollary

Let $\mathcal{X} \subseteq \{0, 1\}^X$ be a closed ID-poset. Then the following are equivalent:

(i) *$\mathcal{X}$ is a lattice ID-poset;*

(ii) *$\mathcal{X}$ is a σ-field.*

Conclusion

To sum up, *ID* is a suitable category in which traditional domains of probability can be characterized by natural properties.

1. *ID*-posets model the sure event, the impossible event, and the negation of an event. States determine events.
2. Closed *ID*-posets satisfy a natural requirement that the domains should be closed with respect to sequential limits.
3. Lattice *ID*-posets are bold algebras.
4. Closed lattice *ID*-posets are Łukasiewicz tribes.
5. The transition from the classical random events represented by σ-fields to fuzzy random events represented by measurable functions is characterized by divisibility.
6. Łukasiewicz tribes form a category in which both classical and fuzzy events live and the probability of an event can be calculated via an integral. From $\mathbf{A}_{\mathcal{X}} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbf{A}_{\mathcal{X}})$ it follows that the classical events (σ-fields of sets) are "minimal" and the fuzzy events (generated Łukasiewicz tribes) are "maximal" probability domains having nice properties.