Migrative Type Functional Equations for Triangular Norms

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Acknowledgement

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- Imre J. Rudas,
- Erich Peter Klement,
- Radko Mesiar.

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Outline

1. Introduction and background
2. Migrative t-norms
   - Migrativity with respect to the minimum
   - Migrativity with respect to strict t-norms
   - Migrativity with respect to nilpotent t-norms
   - Migrativity with respect to a continuous ordinal sum
3. Cross-migrative t-norms
   - Cross-migrativity with respect to the minimum
   - Cross-migrativity with respect to strict t-norms
   - Cross-migrativity with respect to nilpotent t-norms
4. Concluding remarks

INTRODUCTION AND BACKGROUND

Aim of this talk

To deliver results on (continuous) solutions of the following general functional equation ($x, y \in [0,1]$, and $\alpha \in]0,1[$ fixed)

$$\boxed{T_1(T_2(\alpha, x), y) = T_3(x, T_4(\alpha, y)),}$$

where T_1, T_2, T_3, T_4 are triangular norms, in two particular cases:

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- A function $T\colon [0,1]^2 \to [0,1]$ is called associative if it satisfies

$$T(T(x,y),z) = T(x,T(y,z)) \quad \text{for all } x,y,z \in [0,1].$$

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- An example is the product $T_{\mathbf{P}}(x,y) = xy$.

- Let us fix $y = \alpha \in \,]0,1[$. Then we still have

$$T_{\mathbf{P}}(T_{\mathbf{P}}(x,\alpha),z) = T_{\mathbf{P}}(x,T_{\mathbf{P}}(\alpha,z)) \quad \text{for all } x,z \in [0,1].$$

Associativity modified

- Keep $T_{\mathbf{P}}$ inside fixed, and consider a general T outside:

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- Question 1: is there any solution T of the last equations that differs from $T_{\mathbf{P}}$?
- Question 2: what is the link between solutions of the two equations (if any)?

Example

- Consider T_β defined as follows:

$$T_\beta(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ \beta xy & \text{otherwise,} \end{cases}$$

where β is an arbitrary number from $[0,1]$.

- T_β is a t-norm, and it satisfies $T_\beta(\alpha x, y) = T_\beta(x, \alpha y)$ for $(x,y) \in [0,1[$.

- Notice the following particular cases:
  - if $\beta = 0$ then $T_\beta = T_{\mathbf{D}}$ the drastic t-norm;
  - if $\beta = 1$ then $T_\beta = T_{\mathbf{P}}$ the product.

Migrative t-norms

Migrative t-norms

Definition

Let $\alpha \in]0,1[$ and T_1, T_2 be t-norms. We say that the pair (T_1, T_2) is α-migrative (or, equivalently, that T_1 is *α-migrative* with respect to T_2, in symbols $T_1 \underset{\alpha}{\sim} T_2$) if the following functional equation holds:

$$T_1(T_2(\alpha, x), y) = T_1(x, T_2(\alpha, y)) \qquad \text{for all } (x, y) \in [0,1]^2.$$

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$$T_1(T_2(\alpha, x), y) = T_1(x, T_2(\alpha, y)) \qquad \text{for all } (x, y) \in [0,1]^2.$$

Obviously, we have $T_1 \underset{\alpha}{\sim} T_1$ for any t-norm T and for each $\alpha \in\,]0,1[$.

In other words: the relation $\underset{\alpha}{\sim}$ is reflexive on the set of all t-norms.

Equivalent forms of α-migrativity

Theorem

Let α be in $]0,1[$ and T_1, T_2 triangular norms. Then the following statements are equivalent.

(i) (T_1, T_2) *is α-migrative:* $T_1(T_2(\alpha, x), y) = T_1(x, T_2(\alpha, y))$;

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(iii) $T_1(\alpha, x) = T_2(\alpha, x)$ *for all* $x \in [0,1]$.

Equivalent forms of α-migrativity

Theorem

Let α be in $]0,1[$ and T_1, T_2 triangular norms. Then the following statements are equivalent.

(i) (T_1, T_2) *is* α*-migrative:* $T_1(T_2(\alpha, x), y) = T_1(x, T_2(\alpha, y))$;

(ii) (T_2, T_1) *is* α*-migrative:* $T_2(T_1(\alpha, x), y) = T_2(x, T_1(\alpha, y))$;

(iii) $T_1(\alpha, x) = T_2(\alpha, x)$ *for all* $x \in [0, 1]$.

Corollary

The relation $\overset{\alpha}{\sim}$ is an equivalence relation on the set of all t-norms.

Further properties

- If $T_1 \underset{\alpha}{\sim} T_2$ and $]a, b[$ is a non-empty subinterval of $[0, 1]$, and $\alpha \in\,]a, b[$ then for the ordinal sums $(\langle a, b, T_1\rangle)$ and $(\langle a, b, T_2\rangle)$ we have $(\langle a, b, T_1\rangle) \underset{\gamma}{\sim} (\langle a, b, T_2\rangle)$, where $\gamma = \frac{\alpha - a}{b-a}$.

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Recall that for each t-norm T and for each strictly increasing bijection $\varphi\colon [0, 1] \to [0, 1]$ the function $T_\varphi\colon [0, 1]^2 \to [0, 1]$ defined by

$$T_\varphi(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y)))$$

is also a t-norm.

- Let $\varphi\colon [0, 1] \to [0, 1]$ be a strictly increasing bijection, $\alpha \in\,]0, 1[$ and T_1, T_2 be two t-norms. If (T_1, T_2) is α-migrative then $((T_1)_\varphi, (T_2)_\varphi)$ is $\varphi(\alpha)$-migrative.

Cases considered

- We study three particular cases of t-norms that are α-migrative with respect to a fixed T_0:
  - $T_0 = T_{\mathbf{M}}$,
  - $T_0 = T_{\mathbf{P}}$,
  - $T_0 = T_{\mathbf{L}}$.
- Using these results, as a fourth case we study α-migrativity with respect to arbitrary continuous t-norms:
  - $T_0 = (\langle a_i, b_i, T_i \rangle)_{i \in \Gamma}$.

Migrativity with respect to the minimum

Migrativity with respect to $T_{\mathbf{M}}$

Characterization

$$T(\min(\alpha, x), y) = T(x, \min(\alpha, y))$$

Migrativity with respect to $T_{\mathbf{M}}$

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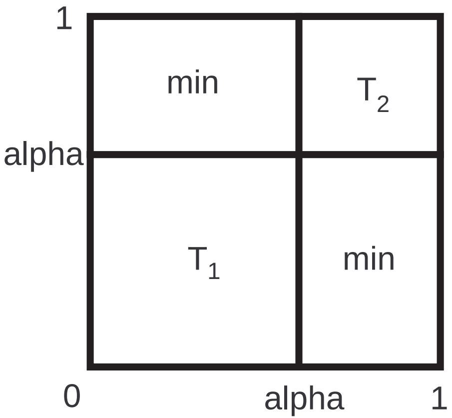
Theorem

A t-norm T is α-migrative with respect to $T_{\mathbf{M}}$ if and only if there exist two t-norms T_1 and T_2 such that T can be written in the following form:

$$T(x,y) = \begin{cases} \alpha T_1\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) & \text{if } x, y \in [0,\alpha], \\ \alpha + (1-\alpha) T_2\left(\frac{x-\alpha}{1-\alpha}, \frac{y-\alpha}{1-\alpha}\right) & \text{if } x, y \in [\alpha, 1], \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Migrativity with respect to T_M

Illustration



Migrativity with respect to strict t-norms

Migrativity with respect to $T_{\mathbf{P}}$

$$T(\alpha x, y) = T(x, \alpha y)$$

Migrativity with respect to $T_{\mathbf{P}}$

$$T(\alpha x, y) = T(x, \alpha y)$$

- Historically, this is the notion introduced originally by Durante and Sarkoczi (2008).
- Rooted in an open problem of the 2nd FSTA, see Mesiar and Novák (1996).

Migrativity with respect to $T_{\mathbf{P}}$

Continuous case, necessary conditions

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Theorem

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Migrativity with respect to $T_{\mathbf{P}}$

Continuous case, necessary conditions

Theorem

If a continuous t-norm T is α-migrative with respect to $T_{\mathbf{P}}$ then T is strict.

If t denotes an additive generator of an α-migrative continuous t-norm T then t satisfies the following functional equation for all $x \in [0, 1]$:

$$t(\alpha x) = t(\alpha) + t(x). \tag{1}$$

Migrativity with respect to $T_{\mathbf{P}}$

Continuous case, characterization and construction

Theorem

Suppose t is an additive generator of a strict t-norm T and α is in $]0,1[$. Then the following statements are equivalent:

(i) *T is α-migrative with respect to $T_{\mathbf{P}}$;*

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Theorem

Suppose t is an additive generator of a strict t-norm T and α is in $]0,1[$. Then the following statements are equivalent:

(i) *T is α-migrative with respect to $T_{\mathbf{P}}$;*

(ii) *t satisfies the functional equation $t(\alpha x) = t(\alpha) + t(x)$;*

(iii) *there exists a continuous, strictly decreasing function t_0 from $[\alpha, 1]$ to the non-negative reals with $t_0(\alpha) < +\infty$ and $t_0(1) = 0$ such that*

$$t(x) = k \cdot t_0(\alpha) + t_0\left(\frac{x}{\alpha^k}\right) \quad \text{if } x \in \left]\alpha^{k+1}, \alpha^k\right], \tag{2}$$

where k is any non-negative integer.

Constructing an additive generator

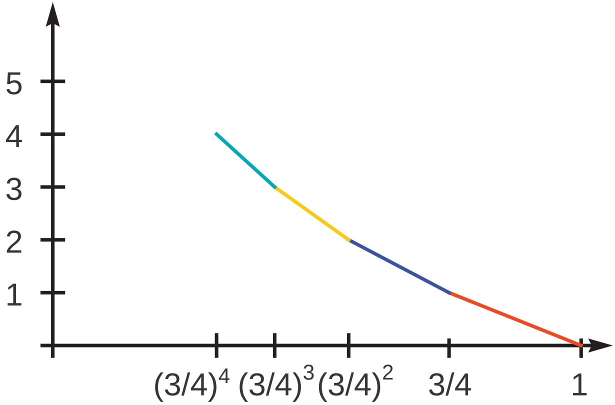
An example

- Let $\alpha = \frac{3}{4}$ and
- $t_0(x) = 4 - 4x$ for $x \in \left[\frac{3}{4}, 1\right]$.
- Then $t\left(\left(\frac{3}{4}\right)^k\right) = k$, and linear in between.

Constructing an additive generator

A graphical illustration

$$t(x) = k \cdot t_0(\alpha) + t_0\left(\frac{x}{\alpha^k}\right) \quad \text{if } x \in \left]\alpha^{k+1}, \alpha^k\right]$$



Migrativity with respect to nilpotent t-norms

Migrativity with respect to $T_{\mathbf{L}}$

Continuous case, necessary condition

$$T(\max(\alpha + x - 1, 0), y) = T(x, \max(\alpha + y - 1, 0))$$

Migrativity with respect to $T_{\mathbf{L}}$

Continuous case, necessary condition

$$T(\max(\alpha + x - 1, 0), y) = T(x, \max(\alpha + y - 1, 0))$$

Theorem

Assume that T is a continuous t-norm that is α-migrative with respect to $T_{\mathbf{L}}$. Then there exists an automorphism φ of the unit interval such that $T = (T_{\mathbf{L}})_{\varphi}$. That is, we have

$$T(x, y) = (T_{\mathbf{L}})_{\varphi}(x, y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1, 0)) \quad \textit{for all } x, y \in [0, 1].$$

Migrativity with respect to $T_{\mathbf{L}}$

Continuous case, characterization and construction

Theorem

Let α be in $]0,1[$ and $n = \max\{k \in \mathbb{N} \mid 1 - k(1-\alpha) > 0\}$.
A t-norm $T(x,y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1, 0))$ $(x, y \in [0,1])$ is α-migrative with respect to $T_{\mathbf{L}}$ if and only if there exist an automorphism ψ_0 of the unit interval with

$$\psi_0\left(\frac{n-(n+1)\alpha}{1-\alpha}\right) = \frac{n-(n+1)\alpha}{1-\alpha}, \qquad (3)$$

such that

Migrativity with respect to $T_{\mathbf{L}}$

Continuous case, characterization and construction

Theorem (cont'd)

$$\varphi(x) = 1 - k(1-\alpha) + (1-\alpha)\psi_0\left(\frac{x - 1 + k(1-\alpha)}{1-\alpha}\right) \quad (4)$$

if $x \in]1 - k(1-\alpha), 1 - (k-1)(1-\alpha)]$ *and* $k \leq n$,

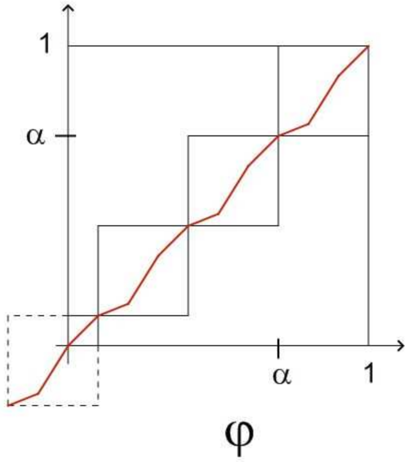
and

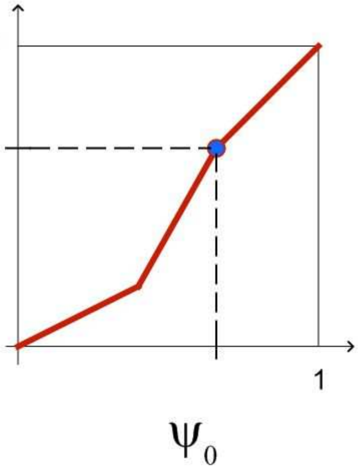
$$\varphi(x) = 1 - (n+1)(1-\alpha) + (1-\alpha)\psi_0\left(\frac{x - 1 + (n+1)(1-\alpha)}{1-\alpha}\right) \quad (5)$$

if $x \in [0, 1 - n(1-\alpha)]$.

Migrativity with respect to T_L

Graphical construction





Migrativity with respect to a continuous ordinal sum

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- We study continuous t-norms T that are α-migrative with respect to a fixed continuous t-norm T_0;
- $\mathcal{T}_{\text{Co}}$: the set of all continuous t-norms;
- $\mathcal{T}_{\text{Ar}}$: the set of all continuous Archimedean t-norms;
- $T = (\langle a_i, b_i, T_i\rangle)_{i\in\Gamma}$, $T_0 = (\langle a_{0j}, b_{0j}, T_{0j}\rangle)_{j\in\Gamma_0}$,

  where $T_i, T_{0j} \in \mathcal{T}_{\text{Ar}}$ for all $i \in \Gamma$ and $j \in \Gamma_0$.
- for any $\alpha \in]0,1[$ there are two exhaustive and mutually exclusive cases:
  - α is an idempotent element of T_0;
  - there exists a $k \in \Gamma$ such that $\alpha \in]a_{0k}, b_{0k}[$.

Migrativity with respect to a continuous ordinal sum

$T_0(\alpha, \alpha) = \alpha$, characterization

Theorem

Suppose T_0 is a continuous t-norm and $\alpha \in]0, 1[$ is an idempotent element of T_0. Then the following statements are equivalent for a continuous t-norm T:

(i) *T is α-migrative with respect to T_0;*

(ii) *T is α-migrative with respect to $T_{\mathbf{M}}$;*

(iii) *there exist continuous t-norms T_1 and T_2 such that T can be written as $T = (\langle 0, \alpha, T_1 \rangle, \langle \alpha, 1, T_2 \rangle)$.*

Migrativity with respect to a continuous ordinal sum

$T_0(\alpha, \alpha) < \alpha$, characterization

Theorem

Suppose $T_0 = (\langle a_{0j}, b_{0j}, T_{0j}\rangle)_{j\in\Gamma_0}$ is a continuous t-norm and $\alpha \in]a_{0k}, b_{0k}[$ for some $k \in \Gamma_0$. Then the following statements are equivalent for a continuous t-norm T.

(i) *T is α-migrative with respect to T_0;*

(ii) *There exist t-norms $T_1, T_3 \in \mathcal{T}_{Co}$ and $T_2 \in \mathcal{T}_{Ar}$ such that*

  (a) *$T = (\langle 0, a_{0k}, T_1\rangle, \langle a_{0k}, b_{0k}, T_2\rangle, \langle b_{0k}, 1, T_3\rangle)$, and*

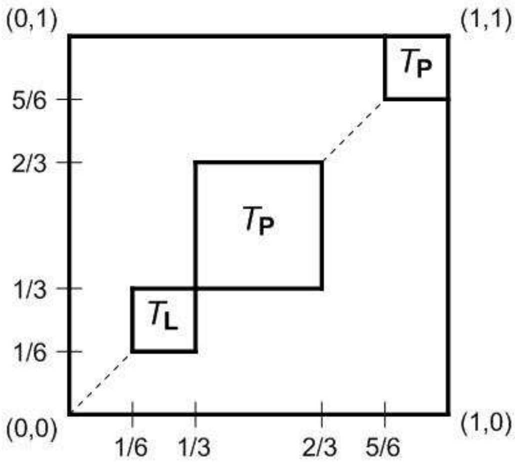
  (b) *T_2 is $\left(\dfrac{\alpha - a_{0k}}{b_{0k} - a_{0k}}\right)$-migrative with respect to T_{0k}.*

Migrativity with respect to a continuous ordinal sum

Remarks

- In the previous theorem $a_{0k} = 0$ (or $b_{0k} = 1$) is possible.
- α-migrativity is restrictive on a continuous T mainly in a neighbourhood of α. This is just α itself if α is an idempotent element of T_0, and it is the square $]a_{0k}, b_{0k}[^2$ otherwise.
- Outside this neighbourhood T can be defined arbitrarily in such a way that the resulting t-norm be continuous.
- The summand $T_2 \in \mathcal{T}_{\mathrm{Ar}}$ in the "middle" of $T = (\langle 0, a_{0k}, T_1\rangle, \langle a_{0k}, b_{0k}, T_2\rangle, \langle b_{0k}, 1, T_3\rangle)$ can be determined by the summand T_{0k} (details in Fodor and Rudas, 2011).

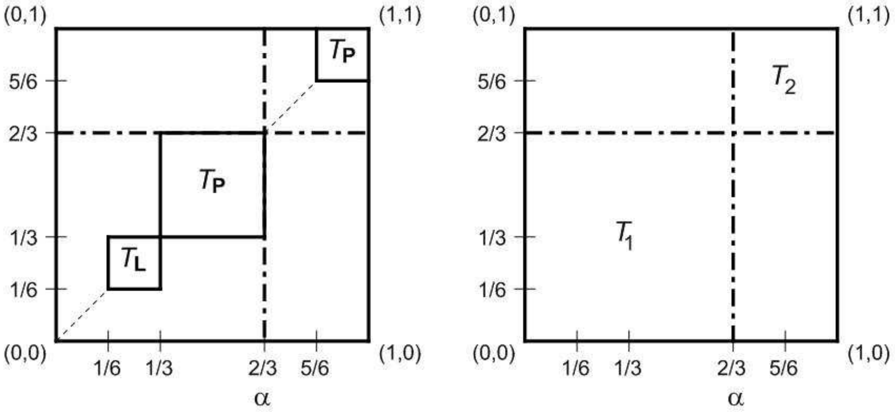
A T_0 for two examples



The ordinal sum $T_0 = (\langle 1/6, 1/3, T_{\mathbf{L}} \rangle, \langle 1/3, 2/3, T_{\mathbf{P}} \rangle, \langle 5/6, 1, T_{\mathbf{P}} \rangle)$.

Example 1

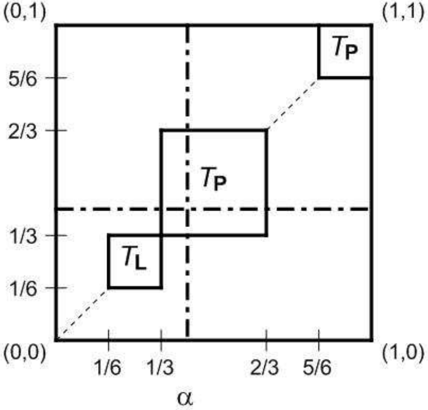
T_0 with $\alpha = 2/3$ (idempotent element)

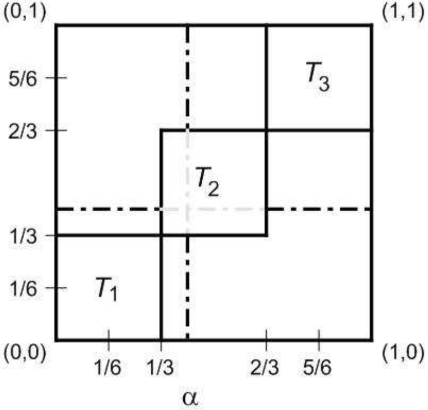


The ordinal sum T_0, and $\alpha = 2/3$ (left). $2/3$-migrative t-norm T with respect to T_0 (right).

Example 2

T_0 with $\alpha = 5/12$ (non-idempotent element)





The ordinal sum T_0, and $\alpha = 5/12$ (left). $5/12$-migrative t-norm T with respect to T_0 (right).

Cross-migrative t-norms

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Definition

Let $\alpha \in]0,1[$ and T_1, T_2 be t-norms. We say that the pair (T_1, T_2) is α-cross-migrative (or, equivalently, that T_1 is *α-cross-migrative* with respect to T_2, in symbols $T_1 \overset{\alpha}{\sim} T_2$) if the following functional equation holds:

$$T_1(T_2(\alpha, x), y) = T_2(x, T_1(\alpha, y)) \qquad \text{for all } (x, y) \in [0,1]^2. \tag{6}$$

Cross-migrative t-norms

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Theorem

The relation $\overset{\alpha}{\sim}$ is reflexive and symmetric on the set of all t-norms.

Cross-migrative t-norms

Further properties

- For each $\alpha \in]0,1[$ and for each t-norm T, $(T, T_{\mathbf{D}})$ is α-cross-migrative.
- The relation $\overset{\alpha}{\sim}$ on the set of t-norms is not transitive and, therefore, no equivalence relation: for each $\alpha \in]0,1[$ we have $T_{\mathbf{M}} \overset{\alpha}{\sim} T_{\mathbf{D}}$ and $T_{\mathbf{D}} \overset{\alpha}{\sim} T_{\mathbf{P}}$, but we do not have $T_{\mathbf{M}} \overset{\alpha}{\sim} T_{\mathbf{P}}$.
- For each t-norm T and for each $c \in [0,1]$, the function $T^{(c)}\colon [0,1]^2 \to [0,1]$ defined by

$$T^{(c)}(x,y) = \begin{cases} T(x,y,c) & \text{if } (x,y) \in [0,1[^2\,, \\ T(x,y) & \text{otherwise,} \end{cases}$$

  is a t-norm (observe that $T^{(0)} = T_{\mathbf{D}}$ and $T^{(1)} = T$), and for each $\alpha \in]0,1[$ we have $T^{(c)} \overset{\alpha}{\sim} T$.

Cross-migrative t-norms

Further properties

- If $T_1 \overset{\alpha}{\sim} T_2$ and $]a,b[$ is a non-empty subinterval of $[0,1]$, then for the ordinal sums $(\langle a, b, T_1\rangle)$ and $(\langle a, b, T_2\rangle)$ we have $(\langle a, b, T_1\rangle) \overset{\gamma}{\sim} (\langle a, b, T_2\rangle)$, where $\gamma = \frac{\alpha - a}{b-a}$.

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Recall that for each t-norm T and for each strictly increasing bijection $\varphi\colon [0, 1] \to [0, 1]$ the function $T_\varphi\colon [0, 1]^2 \to [0, 1]$ defined by

$$T_\varphi(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y)))$$

is also a t-norm.

- Let $\varphi\colon [0, 1] \to [0, 1]$ be a strictly increasing bijection, $\alpha \in]0, 1[$ and T_1, T_2 be two t-norms. If (T_1, T_2) is α-cross-migrative then $((T_1)_\varphi, (T_2)_\varphi)$ is $\varphi(\alpha)$-cross-migrative.

Cross-migrativity with respect to the minimum

Reminder

For a t-norm T and $\alpha \in]0,1[$, $(T, T_{\mathbf{M}})$ being α-cross-migrative means that for all $(x,y) \in [0,1]^2$

$$\boxed{T(\min(\alpha,x),y) = \min(x, T(\alpha,y)).}$$

Cross-migrativity with respect to $T_{\mathbf{M}}$

Characterization - general T

Theorem

Let $\alpha \in \,]0,1[$ and T be a t-norm. Then $(T, T_{\mathbf{M}})$ is α-cross-migrative if and only if there is a $\beta \in [0,\alpha]$ and a t-norm T_1 satisfying

(i) *for all* $(x,y) \in \left[0, \frac{\alpha-\beta}{1-\beta}\right]^2$: $\quad T_1(x,y) = 0,$

(ii) *for all* (x,y) *with* $0 \leq x \leq \frac{\alpha-\beta}{1-\beta} \leq y \leq 1$:

$$T_1(x,y) = \min\left(x, T_1\left(\frac{\alpha-\beta}{1-\beta}, y\right)\right),$$

such that T can be written as

$$T = (\langle \beta, 1, T_1 \rangle). \tag{7}$$

Cross-migrativity with respect to $T_{\mathbf{M}}$

Consequences - general T

- Let $\alpha \in]0,1[$ and T be a t-norm. Then we have:
  - (i) $T(\alpha,\alpha) = \alpha$ and $(T, T_{\mathbf{M}})$ is α-cross-migrative if and only if $(\langle \alpha, 1, T_{\mathbf{D}} \rangle) \leq T$.
  - (ii) If $T(\alpha,\alpha) = \beta < \alpha$ and $(T, T_{\mathbf{M}})$ is α-cross-migrative then $T_*^{(\beta)} \leq T \leq T^*_{(\beta)}$, where the t-norms $T_*^{(\beta)}$ and $T^*_{(\beta)}$ are defined, respectively, by $T_*^{(\beta)} = (\langle \beta, 1, T_{\mathbf{D}} \rangle)$ and

$$T^*_{(\beta)}(x,y) = \begin{cases} \beta & \text{if } (x,y) \in [\beta,\alpha]^2, \\ T_{\mathbf{M}}(x,y) & \text{otherwise.} \end{cases}$$

- The converse is not true.
- These boundaries are sharp because of $T_*^{(\beta)} \overset{\alpha}{\sim} T_{\mathbf{M}}$ and $T^*_{(\beta)} \overset{\alpha}{\sim} T_{\mathbf{M}}$.

Cross-migrativity with respect to $T_{\mathbf{M}}$

Characterization - continuous T

Theorem

Let $\alpha \in \,]0,1[$ and T be a continuous t-norm. Then the following are equivalent:

(i) $(T, T_{\mathbf{M}})$ *is α-cross-migrative.*

(ii) *For all $x \in [0,\alpha]$ we have $T(x,x) = x$, i.e., $T = (\langle \alpha, 1, T_1 \rangle)$ for some continuous t-norm T_1.*

Cross-migrativity with respect to strict t-norms

Reminder

For a t-norm T and $\alpha \in]0,1[$, $(T, T_{\mathbf{P}})$ being α-cross-migrative means that for all $(x, y) \in [0,1]^2$

$$\boxed{T(\alpha x, y) = x\, T(\alpha, y).}$$

We restrict ourselves to continuous solutions only.

Solution

Theorem

Let $\alpha \in \,]0,1[$ *and* T *be a continuous t-norm. Then* $(T, T_{\mathbf{P}})$ *is* α*-cross-migrative if and only if there exist*

- *a* $\beta \in [\alpha, 1]$,
- *a strict t-norm* T_1 *with an additive generator* $t_1 \colon [0,1] \to [0,\infty]$ *satisfying* $t_1(x) = \delta \cdot (d - \log x)$ *for all* $x \in \left[0, \frac{\alpha}{\beta}\right]$ *with some constants* $\delta \in \,]0,\infty[$ *and* $d \in \left]\log \frac{\alpha}{\beta}, \infty\right[$, *and*
- *a continuous t-norm* T_2 *such that*

$$T = (\langle 0, \beta, T_1 \rangle, \langle \beta, 1, T_2 \rangle).$$

Solution rewritten

Theorem

Let $\alpha \in]0,1[$, T_0 *be a strict t-norm with additive generator* $t_0\colon [0,1] \to [0,\infty]$, *and* T *be a continuous t-norm. Then* (T, T_0) *is* α*-cross-migrative if and only if*

$$T = (\langle 0, \beta, T_1\rangle, \langle \beta, 1, T_2\rangle),$$

where $\beta \in [\alpha, 1]$, T_2 *is an arbitrary continuous t-norm and* T_1 *is a strict t-norm with an additive generator* $t_1\colon [0,1] \to [0,\infty]$ *such that there are constants* $d \in \left]-t_0(\frac{\alpha}{\beta}), \infty\right[$ *and* $\delta \in]0,\infty[$ *and we have* $t_1(x) = \delta \cdot (t_0(x) + d)$ *for all* $x \in \left[0, \frac{\alpha}{\beta}\right]$.

Solution in another form

Theorem

Let $\alpha \in]0,1[$, T_0 *be a strict t-norm with additive generator* $t_0\colon [0,1] \to [0,\infty]$, *and* T *be a strict t-norm. Then the following are equivalent:*

(i) (T, T_0) *is* α*-cross-migrative.*

(ii) *The function* $t\colon [0,1] \to [0,\infty]$ *defined by*

$$t(x) = \begin{cases} t_0(x) + c & \text{if } x \in [0,\alpha], \\ t_1(x) & \text{otherwise,} \end{cases} \tag{8}$$

where $c \in]-t_0(\alpha), \infty[$ *and* $t_1\colon [\alpha,1] \to [0,\infty]$ *is a continuous, strictly decreasing function satisfying* $t_1(1) = 0$ *and* $t_1(\alpha) = t_0(\alpha) + c$, *is an additive generator of* T.

Consequences for all strict t-norms

(i) For each $\alpha \in]0,1[$, the relation $\overset{\alpha}{\sim}$ is transitive on the class of all strict t-norms, i.e., $\overset{\alpha}{\sim}$ is an equivalence relation on the class of all strict t-norms.

(ii) For all $\alpha, \beta \in]0,1[$ with $\beta \leq \alpha$ and for all strict t-norms T_1 and T_2 we have that $T_1 \overset{\alpha}{\sim} T_2$ implies $T_1 \overset{\beta}{\sim} T_2$, i.e., the partition of the class of strict t-norms induced by the equivalence relation $\overset{\alpha}{\sim}$ is a refinement of the partition induced by $\overset{\beta}{\sim}$.

(iii) For a fixed $\alpha \in]0,1[$ and a fixed strict t-norm T_0 the equivalence class (with respect to $\overset{\alpha}{\sim}$) $\{T \mid T \text{ is a strict t-norm and } T \overset{\alpha}{\sim} T_0\}$ consists of all strict t-norms satisfying, for some constants $\eta, \vartheta \in]0,1]$ and for all $(x,y) \in [0,\alpha]^2$, the equality $T(x,y,\eta) = T(x,y,\vartheta)$.

Example 1

Consider the family of Dubois-Prade t-norms $(T_\lambda^{\mathbf{DP}})_{\lambda\in[0,1]}$ given by

$$T_\lambda^{\mathbf{DP}} = (\langle 0, \lambda, T_{\mathbf{P}} \rangle).$$

Evidently, for each $\alpha \in]0, \lambda]$ we have $T_\lambda^{\mathbf{DP}} \overset{\alpha}{\sim} T_{\mathbf{P}}$.

Observe that $T_\lambda^{\mathbf{DP}}(x, y, \lambda^2) = T_{\mathbf{P}}(x, y, 1)$ for all $(x, y) \in [0, \alpha]^2$.

Example 2

Consider the Hamacher product $T_{\mathbf{H}}$ which is generated by the additive generator $t_{\mathbf{H}}(x) = \frac{1}{x} - 1$:

$$T_{\mathbf{H}}(x,y) = \frac{x \cdot y}{x + y - x \cdot y}$$

for all $(x,y) \in [0,1]^2 \setminus \{(0,0)\}$).

Define the function $t\colon [0,1] \to [0,\infty]$ by

$$t(x) = \begin{cases} \frac{1}{x} - 1 & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 2 - 2x & \text{otherwise.} \end{cases}$$

Then t is an additive generator of the strict t-norm T given by

Example 2 (cont.)

$$T(x,y)=\begin{cases} T_{\mathbf{H}}(x,y) & \text{if } (x,y)\in \left[0,\frac{1}{2}\right]^2,\\ \frac{x}{1+2x-2x\cdot y} & \text{if } (x,y)\in \left[0,\frac{1}{2}\right[\times\left]\frac{1}{2},1\right],\\ \frac{y}{1+2y-2x\cdot y} & \text{if } (x,y)\in \left]\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right[,\\ \frac{1}{5-2x-2y} & \text{if } (x,y)\in \left]\frac{1}{2},1\right]^2 \text{ and } x+y<\frac{3}{2},\\ x+y-1 & \text{otherwise,} \end{cases}$$

and we have $T \overset{\frac{1}{2}}{\sim} T_{\mathbf{H}}$. Obviously, $T(x,y,1)=T_{\mathbf{H}}(x,y,1)$ for all $(x,y)\in\left[0,\frac{1}{2}\right]^2$.

Cross-migrativity with respect to nilpotent t-norms

Reminder

For a t-norm T and $\alpha \in]0,1[$, $(T, T_{\mathbf{L}})$ being α-cross-migrative means that for all $(x, y) \in [0,1]^2$

$$\boxed{T(\max(x+\alpha-1,0),y) = \max(x+T(\alpha,y)-1,0).}$$

We restrict ourselves to continuous solutions only.

Solution

Theorem

Let $\alpha \in]0,1[$ and T be a continuous t-norm. Then $(T, T_{\mathbf{L}})$ is α-cross-migrative if and only if there exist

- *a $\beta \in [\alpha, 1]$,*
- *a nilpotent t-norm T_1 whose normed additive generator $t_1\colon [0,1] \to [0,\infty]$ satisfies $t_1(x) = 1 - c \cdot x$ for some constant $c \in]0,\infty[$ and all $x \in \left[0, \frac{\alpha}{\beta}\right]$, and*
- *a continuous t-norm T_2 such that*

$$T = (\langle 0, \beta, T_1\rangle, \langle \beta, 1, T_2\rangle).$$

Strict and nilpotent cases differ

- For the product t-norm $T_{\mathbf{P}}$, a continuous t-norm T satisfies $T \overset{\alpha}{\sim} T_{\mathbf{P}}$ if and only if there is a $c \in \left]0, \frac{1}{\alpha}\right[$ such that $T(x, y) = c \cdot x \cdot y$ for all $(x, y) \in [0, \alpha]^2$.
- For the Łukasiewicz t-norm $T_{\mathbf{L}}$, if a continuous t-norm T satisfies $T \overset{\alpha}{\sim} T_{\mathbf{L}}$ then there is a constant $c \in [\alpha, 1]$ such that $T(x, y) = \max(x + y - c, 0)$ for all $(x, y) \in [0, \alpha]^2$.
- The opposite implication may not hold: for the Yager t-norm $T_2^{\mathbf{Y}}$ given by $T_2^{\mathbf{Y}}(x, y) = \max(1 - \sqrt{(1-x)^2 + (1-y)^2}, 0)$ we have $T_2^{\mathbf{Y}}(x, y) = 0 = \max(x + y - 1, 0)$ for all $(x, y) \in \left[0, \frac{1}{5}\right]^2$, but $T_2^{\mathbf{Y}}$ is not $\frac{1}{5}$-cross-migrative with respect to $T_{\mathbf{L}}$.

Characterization

Theorem

Let $\alpha \in]0,1[$, T_0 *be a nilpotent t-norm with additive generator* $t_0\colon [0,1] \to [0,\infty]$, *and* T *be a continuous t-norm. Then* (T, T_0) *is* α*-cross-migrative if and only if*

$$T = (\langle 0, \beta, T_1\rangle, \langle \beta, 1, T_2\rangle),$$

where $\beta \in [\alpha, 1]$, T_2 *is an arbitrary continuous t-norm and* T_1 *is a nilpotent t-norm with an additive generator* $t_1\colon [0,1] \to [0,\infty]$ *such that there are constants* $d \in \left]-t_0(\frac{\alpha}{\beta}), \infty\right[$ *and* $\delta \in]0,\infty[$ *and we have* $t_1(x) = \delta \cdot (t_0(x) + d)$ *for all* $x \in \left[0, \frac{\alpha}{\beta}\right]$.

Consequences

- For each $\alpha \in]0,1[$, the relation $\overset{\alpha}{\sim}$ is transitive on the class of all nilpotent t-norms, i.e., $\overset{\alpha}{\sim}$ is an equivalence relation on the class of all nilpotent t-norms.
- For all $\alpha, \beta \in]0,1[$ with $\beta \leq \alpha$ and for all nilpotent t-norms T_1 and T_2 we have that $T_1 \overset{\alpha}{\sim} T_2$ implies $T_1 \overset{\beta}{\sim} T_2$, i.e., the partition of the class of nilpotent t-norms induced by the equivalence relation $\overset{\alpha}{\sim}$ is a refinement of the partition induced by $\overset{\beta}{\sim}$.

Concluding remarks

Remarks 1

- Migrativity ($T_1(T_2(\alpha,x),y) = T_1(x, T_2(\alpha,y))$) and cross-migrativity ($T_1(T_2(\alpha,x),y) = T_2(x, T_1(\alpha,y))$) of t-norms are interesting properties expressed in the form of functional equations.
- We have given characterizations for the basic continuous Archimedean t-norms and for the minimum.
- Constructions could be illustrated graphically. This also supports the term "migrative" (characterized by migration; undergoing periodic migration).
- While migrativity defines an equivalence relation on the set of t-norms, cross-migrativity implies an equivalence relation only in the classes of strict and nilpotent t-norms.

Remarks 2

- The α-cross-migrativity can be seen as a special kind of *commuting* of t-norms T_1 and T_2, if we rewrite the equation into the equivalent form

$$T_1(T_2(x,\alpha), T_2(1,y)) = T_2(T_1(x,1), T_1(\alpha,y)).$$

It seems to be interesting to study the functional equation (valid for all $(x,y) \in [0,1]^2$)

$$T_1(T_2(x,\alpha), T_2(\beta,y)) = T_2(T_1(x,\beta), T_1(\alpha,y)).$$

THANK YOU FOR YOUR ATTENTION!