Rough sets generated by a pair of monoidal type structures

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- Rough sets (Pawlak 1983)
- Soft sets (Molodtsov 1999)
- Fuzzy rough sets, rough fuzzy sets, etc (Dubois, Prade, Pavlowski, Yao, et al.)

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  5. Approximate systems

Monoidal type structure

A complete infinitely distributive lattice

$$(L, \leq, \wedge, \vee)$$

with the smallest and the largest elements 0_L and 1_L respectively.
$*, \odot : L \times L \to L$ are commutative associative monotone operations on L, distributing over arbitrary joins, and $1_L * \alpha = \alpha$, $1_L \odot \alpha = \alpha$ for every $\alpha \in L$.
There is a further binary operation - residium $\mapsto$ on a lattice

$$\alpha \mapsto \beta = \bigvee\{\gamma \mid \gamma \odot \alpha \leq \beta, \gamma \in L\}$$

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2. The family of all L-sets on X is denoted by L^X.

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3. symmetric if $\rho(x, y) = \rho(y, x)$,
4. transitive if $\rho(x, y) * \rho(y, z) \le \rho(x, z)$ for all $x, y, z \in X$.

Category of sets with L-relations

Let L be fixed. Given sets equipped with L-relations (X, ρ), (Y, σ), we consider mappings $f : X \to Y$ respecting this relations:

$$\sigma(f(x), f(x')) \geq \rho(x, x') \quad \forall x, x' \in X$$

In the result we obtain a category **REL**(L).
This and some related categories were studied by different authors.

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1. We construct a lower approximation operator: $l_\rho \colon L^X \to L^X$

$$l_\rho(A)(x) = \inf_{x' \in X} (\rho(x, x') \mapsto A(x')).$$

$$\alpha \mapsto \beta = \bigvee \{\gamma \mid \gamma \odot \alpha \leq \beta, \gamma \in L\}$$

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3. We call the triple $(A, l_\rho(A), u_\rho(A))$ an L-rough set and study its properties.

Example: Classical rough sets

In case $L = \{0, 1\}$ we obtain Pawlak model of rough sets:

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6. The triple $(A, l_\rho(A), u_\rho(A))$ is a rough set in Pawlak's sense.

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6. The triple $(A, l_\rho(A), u_\rho(A))$ is an L-rough set defined with Łukasiewicz t-norm.

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3. Let product t-norm $T_P(a, b) = a \cdot b$ is used for lower approximation and Łukasiewicz t-norm $T_L(a, b) = max(a + b - 1, 0)$ be used for upper approximation.

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4. Define a lower approximation of A:

$$l_\rho(A)(x) = \inf_{x' \in X} \begin{cases} 1, \rho(x, x') \leq A(x') \\ A(x')/\rho(x, x'), \rho(x, x') > A(x') \end{cases}$$

Example: L-rough sets with Łukasiewicz t-norm and product t-norm II

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Example: L-rough sets with Łukasiewicz t-norm and product t-norm II

- Define an upper approximation of A:

$$u_\rho(A)(x) = \sup_{x' \in X} (max(\rho(x, x') + A(x') - 1, 0)).$$

- The triple $(A, l_\rho(A), u_\rho(A))$ is an L-rough set, where the upper approximation is obtained with Łukasiewicz t-norm and the lower approximation is obtained by means of the product t-norm.

Approximate system I

The concept of an approximate system was introduced in

A.Šostak, On approximative fuzzy operators, 1^{st} Czech-Latvian Seminar on Fuzzy Sets and Soft Computing, 2008 Trojanice, Czech Republic, Abstracts 7-8

and further studied in

A.Šostak, Towards the theory of M-approximate systems: Fundamentals and examples, Fuzzy Sets and Syst. **161** (2010), 2440 - 2461.

Approximate systems make a common background for describing and studying fuzzy sets, (fuzzy) topological structures and (fuzzy) rough sets.

Approximate system II

Definition

Given a lattice L; a pair of mappings $u, l : L^X \to L^X$ is called an approximation operators if they satisfying the following conditions:

1. $l(1_L) = 1_L$;
2. $a \geq l(a) \ \forall a \in L^X$;
3. $l(a \wedge b) = l(a) \wedge l(b) \ \forall a, b \in L^X$;
4. $l(l(a)) = l(a) \ \forall a \in L^X$.

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4. $l(l(a)) = l(a)\ \forall a \in L^X$.
5. $u(0_L) = 0_L$;
6. $a \leq u(a)\ \forall a \in L^X$
7. $u(a \vee b) = u(a) \vee u(b)\ \forall a, b \in L^X$;
8. $u(u(a)) = u(a)\ \forall a \in L^X$.

Approximate system III

In this case $l : L^X \to L^X$ and $u : L^X \to L^X$ are called respectively an upper approximation operator and a lower approximation operators on the lattice L. The triple (L^X, l, u) is approximate system.

Approximate system as L-rough sets

Theorem

If L-relation ρ is transitive and reflexive then L-rough set $(A, l_\rho(A), u_\rho(A))$, where $A \in L^X$ determines an approximate system $(L^X, l_\rho(A), u_\rho(A))$.

Lattice of approximate structures on fixed lattice L

Given two approximate systems (l, u) and (l', u') we say $(l, u) \preceq (l', u')$ iff $l \leq l'$ and $u \geq u'$.

L-rough systems as approximate systems

Let **AR** (L) be the family of all L-rough sets generated by L-relations.
We introduce an oder on the family **AR** (L) by pointwise extending it from the order of lattice L:

$$\rho \leq \sigma \iff \rho(x, x') \leq \sigma(x, x') \text{ for every } \; x, x' \in X$$

Theorem

Given two L-relations $\rho, \sigma : X \times X \to L$ on a set X

$$\rho \leq \sigma \iff (l_\rho, u_\rho) \succeq (l_\sigma, u_\sigma).$$

Lattice structure of **ASR**

ASR (L) be subfamily of **AR** (L) generated by L-reflexive and transitive relations on X.

Theorem

Let a family $\mathcal{R} = \{\rho_i \mid i \in I\}$ of reflexive, transitive L-relations on a set X be given. Then

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1.
$$l_{\bigwedge_{i\in I}\rho_i}(A)(x) \geq \bigwedge_{i\in I} l_{\rho_i}(A)(x),$$
$$u_{\bigwedge_{i\in I}\rho_i}(A)(x) \leq \bigwedge_{i\in I} u_{\rho_i}(A)(x), \quad \forall A \in L^X,\ \forall x \in X;$$

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2.
$$l_{\vee_{i\in I}\rho_i}(A)(x) = \bigwedge_{i\in I} l_{\rho_i}(A)(x)$$
$$u_{\vee_{i\in I}\rho_i}(A)(x) = \bigvee_{i\in I} u_{\rho_i}(A)(x) \quad \forall A \in L^X,\ \forall x \in X$$

Lattice structure of **ASR** II

Theorem

There is a isomorphism between the lattice of the transitive and reflexive L-relations $R = \{\rho : X \times X \to L\}$ *and the lattice of the approximation systems defined by this relation and, respectively, corresponding L-rough sets.*
$A_R = \{(l, u) \mid l\colon L^X \to L^X; u\colon L^X \to L^X\}$.
This isomorphism is given by

$$(X, \rho) \longmapsto (L^X, l_\rho, u_\rho).$$

Approximate system with different t-norms

Theorem

Given t-norms $*_1$,$*_2$, $\odot_1$, $\odot_2$ *an* L*-relation* ρ *on* X*.*
If $*_1 \leq *_2$ *and* $\odot_1 \leq \odot_2$ *then it means that* $l_\rho(A, \odot_1) \geq l_\rho(A, \odot_2)$ *and* $u_\rho(A, *_1) \leq u_\rho(A, *_2)$ *for every* $A \in L^X$*.*

Thank you for your attention!

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