State-morphism MV-algebras and Their Generalizations

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ERDF OP R&D Projects CE meta-QUTE ITMS 26240120022.
$M$ - MV-algebra, we define a partial operation $+$, via $a + b$ is defined iff $a \leq b^*$ iff $a \circ b = 0$, then $a + b := a \oplus b$. 
• $M$ - MV-algebra, we define a partial operation $+$, via $a + b$ is defined iff $a \leq b^*$ iff $a \odot b = 0$, then $a + b := a \oplus b$.

• $+$ restriction of the $\ell$-group addition
States on MV-algebras

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- $+$ restriction of the $\ell$-group addition
- state- $s : M \rightarrow [0, 1]$, (i) $s(a + b) = s(a) + s(b)$, (ii) $s(1) = 1$. 
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• $S(M)$ -set of states. $S(M) \neq \emptyset$. 
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• state- $s : M \rightarrow [0, 1]$, (i) $s(a + b) = s(a) + s(b)$,
(ii) $s(1) = 1$.

• $S(M)$ -set of states. $S(M) \neq \emptyset$.

• extremal state $s = \lambda s_1 + (1 - \lambda) s_2$ for
$\lambda \in (0, 1) \Rightarrow s = s_1 = s_2$. 


\{ s_\alpha \} \rightarrow s \iff \lim_\alpha s_\alpha(a) \rightarrow s(a), \ a \in M.
\[ \{s_\alpha\} \rightarrow s \text{ iff } \lim_\alpha s_\alpha(a) \rightarrow s(a), \ a \in M. \]

- \( S(E) \) - Hausdorff compact topological space,
- \( \partial_e S(M) \)
• \( \{ s_{\alpha} \} \to s \) iff \( \lim_{\alpha} s_{\alpha}(a) \to s(a), \ a \in M. \)

• \( \mathcal{S}(E) \) - Hausdorff compact topological space, \( \partial_e \mathcal{S}(M) \)

• Krein-Mil’man \( \mathcal{S}(M) = \text{Cl}(\text{ConHul}(\partial_e \mathcal{S}(M))) \)
\begin{itemize}
  \item \{s_\alpha\} \rightarrow s \text{ iff } \lim_\alpha s_\alpha(a) \rightarrow s(a), \; a \in M.
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  \item \text{Krein-Mil'man } \mathcal{S}(M) = \text{Cl}(\text{ConHul}(\partial_e \mathcal{S}(M))
  \item s \text{ is extremal iff } s(a \land b) = \min\{s(a), s(b)\} \iff s \text{ is MV-homomorphism iff } \text{Ker}(s) \text{ is a maximal ideal.}
\end{itemize}
\[ \{s_\alpha\} \rightarrow s \text{ iff } \lim_{\alpha} s_\alpha(a) \rightarrow s(a), \ a \in M. \]

- \( S(E) \) - Hausdorff compact topological space, \( \partial_e S(M) \)
- Krein-Mil'man \( S(M) = \text{Cl}(\text{ConHul}(\partial_e S(M))) \)
- \( s \) is extremal iff \( s(a \wedge b) = \min\{s(a), s(b)\} \) iff \( s \) is MV-homomorphism iff \( \text{Ker}(s) \) is a maximal ideal.
- \( s \leftrightarrow \text{Ker}(s), \ 1-1 \) correspondence
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• Kernel-hull topology = $\partial_e S(E)$ set of extremal states
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• Kroupa–Panti $a \mapsto \hat{a}, \hat{a}(s) := s(a), \\
s(a) = \int_{\partial_e S(M)} \hat{a}(t) d\mu_s(t)$
• every maximal ideal is a kernel of a unique state

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• Kroupa- Panti $a \mapsto \hat{a}, \hat{a}(s) := s(a),$

\[
s(a) = \int_{\partial_e S(M)} \hat{a}(t) d\mu_s(t)
\]

• $\mu_s$ - unique Borel regular $\sigma$-additive probability measure on $\mathcal{B}(S(M))$ such that $\mu_s(\partial_e S(M)) = 1$. 
State MV-algebras

- MV-algebras with a state are not universal algebras, and therefore, they do not provide an algebraizable logic for probability reasoning over many-valued events.
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- Flaminio-Montagna - introduce an algebraizable logic whose equivalent algebraic semantics is the variety of state MV-algebras.
State MV-algebras

- MV-algebras with a state are not universal algebras, and therefore, they do not provide an algebraizable logic for probability reasoning over many-valued events.
- Flaminio-Montagna - introduce an algebraizable logic whose equivalent algebraic semantics is the variety of state MV-algebras.
- A state MV-algebra is a pair $(M, \tau)$, $M$ - MV-algebra, $\tau$ unary operation on $A$ s.t.
• \( \tau(1) = 1 \)
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• \( \tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \otimes y)) \)
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\[ \tau(x^*) = \tau(x)^* \]
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• \( \tau(x^*) = \tau(x)^* \)

• \( \tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y) \)

• \( \tau \) -internal operator, state operator
Properties

- \( \tau^2 = \tau \)
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- \( \tau(x \odot y) = \tau(x) \odot \tau(y) \) if \( x \odot y = 0 \).
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Properties

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• if \( (M, \tau) \) is s.i., then \( \tau(M) \) is a chain

• if \( (M, \tau) \) is s.i., then \( M \) is not necessarily a chain
- \( F \)-filter, \( \tau \)-filter if \( \tau(F) \subseteq F \).
• $F$-filter, $\tau$-filter if $\tau(F) \subseteq F$.
• 1-1 correspondence congruences and $\tau$-filters
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• 1-1 correspondence congruences and $\tau$-filters

• $M = [0, 1] \times [0, 1]$, $\tau(x, y) = (x, x)$ s.i. - not chain
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- 1-1 correspondence congruences and $\tau$-filters
- $M = [0, 1] \times [0, 1]$, $\tau(x, y) = (x, x)$ s.i. - not chain
- state-morphism $(M, \tau)$, $\tau$ is an idempotent endomorphism
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• 1-1 correspondence congruences and \( \tau \)-filters

• \( M = [0, 1] \times [0, 1], \tau(x, y) = (x, x) \) s.i. - not chain

• state-morphism \( (M, \tau) \), \( \tau \) is an idempotent endomorphism

• \( s \) state on \( M, [0, 1] \otimes M \),
\[ \tau_s(\alpha \otimes a) := \alpha \cdot s(a) \otimes 1 \]
\( ([0, 1] \otimes, \tau_s) \) is an SMV-algebra.
• $([0, 1] \otimes, \tau_s)$ is an SMV-algebra.

• $([0, 1] \otimes, \tau_s)$ is an SMMV-algebra iff $s$ is an extremal state.
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• if \(M\) is a chain, every SMV-algebra \((M, \tau)\) is an SMMV-algebra
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• if \(M\) is a chain, every SMV-algebra \((M, \tau)\) is an SMMV-algebra

• if \(\tau(M) \in V(S_1, \ldots, S_n)\) for some \(n \geq 1\), then \((M, \tau)\) is an SMMV-algebra
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• if \(\tau(M) \in V(S_1, \ldots, S_n)\) for some \(n \geq 1\), then \((M, \tau)\) is an SMMV-algebra

• Iff \(\tau((n + 1)x) = \tau(nx)\)
Other examples

Let \((G, u)\) be a unital Riesz space and let 
\[ A = \Gamma(G, u) \]. Choose an endomorphism 
\[ s : A \to A \] such that \( s \circ s = s \) and fix a real 
number \( \lambda \in [0, 1] \). Define a mapping 
\[ s_\lambda : A^2 \to A \] by

\[ s_\lambda(x, y) = \lambda s(x) + (1 - \lambda) s(y), \quad (x, y) \in A^2, \]

and set \( \tau_{\lambda, s} : A^3 \to A^3 \) by

\[ \tau_{\lambda, s}(x, y, z) = (s_\lambda(x, y), s_\lambda(x, y), s_\lambda(x, y)), \quad (x, y, z) \in A^3. \]
Then \( D(A)_{\lambda,s} := (A^3, \tau_{\lambda,s}) \) is a state MV-algebra such that it is a state morphism MV-algebra iff \( \lambda \in \{0, 1\} \).

\[
\text{Ker}(\tau_{\lambda,s}) = \begin{cases} 
\text{Ker}(s) \times \text{Ker}(s) \times A & \text{if } \lambda \in (0, 1), \\
A \times \text{Ker}(s) \times A & \text{if } \lambda = 0, \\
\text{Ker}(s) \times A \times A & \text{if } \lambda = 1.
\end{cases}
\]

Hence, if \( A \) is a subdirectly irreducible MV-algebra and \( s \) is faithful, then neither \((A, \tau_{0,s})\) nor \((A, \tau_{1,s})\) is a subdirectly irreducible state morphism MV-algebra.
whilst if $\lambda \in (0, 1)$, $D(A_{\lambda,s})$ is a subdirectly irreducible state MV-algebra.
• whilst if $\lambda \in (0, 1)$, $D(A_{\lambda,s})$ is a subdirectly irreducible state MV-algebra.

• Let $A = [0, 1]$ be the standard MV-algebra and let $s$ be the identity on $A$. If $\lambda \in \{0, 1\}$, then $D([0, 1]_{1,s})$ generates the variety of state morphism MV-algebras.
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• Let $A = [0, 1]$ be the standard MV-algebra and let $s$ be the identity on $A$. If $\lambda \in \{0, 1\}$, then $D([0, 1]_{1,s})$ generates the variety of state morphism MV-algebras.

• Does $D([0, 1]_{\lambda,s})$ for a fixed $\lambda \in (0, 1)$ and the identity $s$ on $[0, 1]$ generate the variety of state MV-algebras?
State BL-algebras

- $M$ - BL-algebra. A map $\tau : M \to M$ s.t.

  1. $\tau(0) = 0$;
  2. $\tau(x \rightarrow y) = \tau(x) \rightarrow \tau(x \land y)$;
  3. $\tau(x \odot y) = \tau(x) \odot \tau(x \rightarrow (x \odot y))$;
  4. $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y)$;
  5. $\tau(\tau(x) \rightarrow \tau(y)) = \tau(x) \rightarrow \tau(y)$

  state-operator on $M$, pair $(M, \tau)$ - state BL-algebra
State BL-algebras

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1. $\tau(0) = 0$;
2. $\tau(x \rightarrow y) = \tau(x) \rightarrow \tau(x \land y)$;
3. $\tau(x \circ y) = \tau(x) \circ \tau(x \rightarrow (x \circ y))$;
4. $\tau(\tau(x) \circ \tau(y)) = \tau(x) \circ \tau(y)$;
5. $\tau(\tau(x) \rightarrow \tau(y)) = \tau(x) \rightarrow \tau(y)$

state-operator on $M$, pair $(M, \tau)$ - state BL-algebra

- If $\tau : M \rightarrow M$ is a BL-endomorphism s.t. $\tau \circ \tau = \tau$, - state-morphism operator and the couple $(M, \tau)$ - state-morphism BL-algebra.
- every state operator on a linear BL-algebra is a state-morphism
• every state operator on a linear BL-algebra is a state-morphism

**Example 0.2** Let \( M \) be a BL-algebra. On \( M \times M \) we define two operators, \( \tau_1 \) and \( \tau_2 \), as follows

\[
\tau_1(a, b) = (a, a), \quad \tau_2(a, b) = (b, b), \quad (a, b) \in M \times M.
\]

Then \( \tau_1 \) and \( \tau_2 \) are two state-morphism operators on \( M \times M \).
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**Example 0.3** Let $M$ be a BL-algebra. On $M \times M$ we define two operators, $\tau_1$ and $\tau_2$, as follows

$$\tau_1(a, b) = (a, a), \quad \tau_2(a, b) = (b, b), \quad (a, b) \in M \times M. \quad (2.0)$$

Then $\tau_1$ and $\tau_2$ are two state-morphism operators on $M \times M$.

$\text{Ker}(\tau) = \{a \in M : \tau(a) = 1\}$. 
We say that two subhoops, $A$ and $B$, of a BL-algebra $M$ have the *disjunction property* if for all $x \in A$ and $y \in B$, if $x \lor y = 1$, then either $x = 1$ or $y = 1$. 
• We say that two subhoops, $A$ and $B$, of a BL-algebra $M$ have the disjunction property if for all $x \in A$ and $y \in B$, if $x \lor y = 1$, then either $x = 1$ or $y = 1$.

• **Lemma 0.5** Suppose that $(M, \tau)$ is a state BL-algebra. Then:

(1) *If $\tau$ is faithful, then $(M, \tau)$ is a subdirectly irreducible state BL-algebra if and only if $\tau(M)$ is a subdirectly irreducible BL-algebra.*

Now let $(M, \tau)$ be subdirectly irreducible. Then:
• (2) \( \text{Ker}(\tau) \) is (either trivial or) a subdirectly irreducible hoop.

(3) \( \text{Ker}(\tau) \) and \( \tau(M) \) have the disjunction property.
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• (3) $\text{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.

**Theorem 0.7** Let $(M, \tau)$ be a state BL-algebra satisfying conditions (1), (2) and (3) in the last Lemma. Then $(M, \tau)$ is subdirectly irreducible.
• **Theorem 0.8** A state-morphism BL-algebra \((M, \tau)\) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.
Theorem 0.9 A state-morphism BL-algebra $(M, \tau)$ is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.

(i) $M$ is linear, $\tau = \text{id}_M$, and the BL-reduct $M$ is a subdirectly irreducible BL-algebra.
Theorem 0.10  A state-morphism BL-algebra \((M, \tau)\) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.

(i) \(M\) is linear, \(\tau = \text{id}_M\), and the BL-reduct \(M\) is a subdirectly irreducible BL-algebra.

(ii) The state-morphism operator \(\tau\) is not faithful, \(M\) has no nontrivial Boolean elements, and the BL-reduct \(M\) of \((M, \tau)\) is a local BL-algebra, \(\text{Ker}(\tau)\) is a subdirectly irreducible irreducible hoop, and \(\text{Ker}(\tau)\) and \(\tau(M)\) have the disjunction property.
Theorem 0.11 A state-morphism BL-algebra $(M, \tau)$ is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.

(i) $M$ is linear, $\tau = \text{id}_M$, and the BL-reduct $\overline{M}$ is a subdirectly irreducible BL-algebra.

(ii) The state-morphism operator $\tau$ is not faithful, $M$ has no nontrivial Boolean elements, and the BL-reduct $\overline{M}$ of $(M, \tau)$ is a local BL-algebra, $\text{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\text{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.
Theorem 0.12 A state-morphism BL-algebra \((M, \tau)\) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.

(i) \(M\) is linear, \(\tau = \text{id}_M\), and the BL-reduct \(M\) is a subdirectly irreducible BL-algebra.

(ii) The state-morphism operator \(\tau\) is not faithful, \(M\) has no nontrivial Boolean elements, and the BL-reduct \(M\) of \((M, \tau)\) is a local BL-algebra, \(\text{Ker}(\tau)\) is a subdirectly irreducible irreducible hoop, and \(\text{Ker}(\tau)\) and \(\tau(M)\) have the disjunction property.
Theorem 0.13 A state-morphism BL-algebra $(M, \tau)$ is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.

(i) $M$ is linear, $\tau = \text{id}_M$, and the BL-reduct $M$ is a subdirectly irreducible BL-algebra.

(ii) The state-morphism operator $\tau$ is not faithful, $M$ has no nontrivial Boolean elements, and the BL-reduct $M$ of $(M, \tau)$ is a local BL-algebra, $\text{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\text{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.
Moreover, $M$ is linearly ordered if and only if $\text{Rad}_1(M)$ is linearly ordered, and in such a case, $M$ is a subdirectly irreducible BL-algebra such that if $F$ is the smallest nontrivial state-filter for $(M, \tau)$, then $F$ is the smallest nontrivial BL-filter for $M$. 
Moreover, $M$ is linearly ordered if and only if $\text{Rad}_1(M)$ is linearly ordered, and in such a case, $M$ is a subdirectly irreducible BL-algebra such that if $F$ is the smallest nontrivial state-filter for $(M, \tau)$, then $F$ is the smallest nontrivial BL-filter for $M$.

If $\text{Rad}(M) = \text{Ker}(\tau)$, then $M$ is linearly ordered.
(iii) The state-morphism operator $\tau$ is not faithful, $M$ has a nontrivial Boolean element. There are a linearly ordered BL-algebra $A$, a subdirectly irreducible BL-algebra $B$, and an injective BL-homomorphism $h : A \to B$ such that $(M, \tau)$ is isomorphic as a state-morphism BL-algebra with the state-morphism BL-algebra $(A \times B, \tau_h)$, where

$$\tau_h(x, y) = (x, h(x))$$

for any $(x, y) \in A \times B$. 
Varieties of SMMV-algebras

- Komori - countably many subvarieties of MV-algebras
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- $\mathcal{V}$-variety of MV-algebras, $\mathcal{V}_\tau$ -system of SMMV-algebras $(M, \tau)$ s.t $M \in \mathcal{V} \in \mathcal{V}$. 
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- \( D(M) := (M \times M, \tau_M) \)
- \( \mathcal{V}(D) = \mathcal{V}(M)_\tau \)
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- $D(M) := (M \times M, \tau_M)$
- $\mathcal{V}(D) = \mathcal{V}(M)_\tau$
- $SMMV = \mathcal{V}(D([0, 1]))$
Varieties of SMMV-algebras

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- $D(M) := (M \times M, \tau_M)$
- $\mathcal{V}(D) = \mathcal{V}(M)_\tau$
- $\mathcal{SMMV} = \mathcal{V}(D([0, 1]))$
- $\mathcal{P}_\tau = \mathcal{V}(D(C))$, $\mathcal{P}$ perfect MV-algebras, $C$ - Chang
Theorem: $\mathcal{VI} \subseteq \mathcal{VR} \subseteq \mathcal{VL} \subseteq \mathcal{V}_\tau$. and all inclusions are proper of $\mathcal{V}$ is not finitely generated.
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Theorem: Representable SMMV-algebras:

$$\tau(x) \lor (x \rightarrow (\tau(y) \leftrightarrow y)) = 1.$$
• Theorem: $\forall I \subseteq \forall R \subseteq \forall L \subseteq \forall \tau$. and all inclusions are proper of $\forall$ is not finitely generated.

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Theorem: Representable SMMV-algebras:

$$\tau(x) \lor (x \rightarrow (\tau(y) \leftrightarrow y)) = 1.$$ 

also for BL-algebra

Theorem: $\forall L$ - generated by those $(M, \tau), M$ is local

$$(\tau(x) \leftrightarrow x)^* \leq (\tau(x) \leftrightarrow x).$$
Uncountable many subvarieties

- $[0, 1]^* \text{ ultrapower, for positive infinitesimal } \\
  \epsilon \in [0, 1]^*$
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- $X$ subset of prime numbers, $A(X)$ MV-algebra generated by $\epsilon$ and $\frac{n}{m}$ s.t
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- (1) either $n = 0$ or $\text{g.c.d}(n, m) = 1$
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* $\forall p \in X$, $p$ does not divide $m$
Uncountable many subvarieties

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- \(X\) subset of prime numbers, \(A(X)\) MV-algebra generated by \(\epsilon\) and \(\frac{n}{m}\) s.t
- (1) either \(n = 0\) or g.c.d\((n, m) = 1\)
- \(\forall p \in X\), \(p\) does not divide \(m\)
- \(\tau(x) = \) standard part of \(x\)
Uncountable many subvarieties

- \([0, 1]^*\) ultrapower, for positive infinitesimal \(\epsilon \in [0, 1]^*\)
- \(X\) subset of prime numbers, \(A(X)\) MV-algebra generated by \(\epsilon\) and \(\frac{n}{m}\) s.t
  - (1) either \(n = 0\) or \(\text{g.c.d}(n, m) = 1\)
  - \(\forall p \in X, p\) does not divide \(m\)
  - \(\tau(x) =\) standard part of \(x\)
  - \((A(X), \tau)\) is linearly ordered SMMV-algebra
• if $X \neq Y$, then $V(A(X)) \neq V(A(Y))$
• if $X \neq Y$, then $V(A(X)) \neq V(A(Y))$

• Theorem: Between $\mathcal{MVI}$ and $\mathcal{MVR}$ there is uncountably many varieties
Generators of SMBL-algebras

- \textit{t-norm} function \( t : [0, 1] \times [0, 1] \rightarrow [0, 1] \) such that (i) \( t \) is commutative, associative, (ii) \( t(x, 1) = x, x \in [0, 1] \), and (iii) \( t \) is nondecreasing.
Generators of SMBL-algebras

- **t-norm** function $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that (i) $t$ is commutative, associative, (ii) $t(x, 1) = x$, $x \in [0, 1]$, and (iii) $t$ is nondecreasing.

- If $t$ is continuous, we define $x \ominus_t y = t(x, y)$ and $x \rightarrow_t y = \sup\{z \in [0, 1] : t(z, x) \leq y\}$ for $x, y \in [0, 1]$, then $I_t := ([0, 1], \min, \max, \ominus_t, \rightarrow_t, 0, 1)$ is a BL-algebra.
Generators of SMBL-algebras

- **t-norm**- function $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that (i) $t$ is commutative, associative, (ii) $t(x, 1) = x$, $x \in [0, 1]$, and (iii) $t$ is nondecreasing.

- If $t$ is continuous, we define $x \odot_t y = t(x, y)$ and $x \rightarrow_t y = \sup\{z \in [0, 1] : t(z, x) \leq y\}$ for $x, y \in [0, 1]$, then $\mathbb{I}_t := ([0, 1], \min, \max, \odot_t, \rightarrow_t, 0, 1)$ is a BL-algebra.

- The variety of all BL-algebras is generated by all $\mathbb{I}_t$ with a continuous t-norm $t$. 
\( \mathcal{T} \) denotes the system of all BL-algebras 
\( D(\mathbb{I}_t) \), where \( t \) is a continuous t-norm on the 
interval \([0, 1]\),
• $\mathcal{T}$ denotes the system of all BL-algebras $D(\mathbb{I}_t)$, where $t$ is a continuous t-norm on the interval $[0, 1]$,

• **Theorem 0.15** *The variety of all state-morphism BL-algebras is generated by the class $\mathcal{T}$.*
General Approach - State-Morphism Algebras

- $A$ an algebra of type $F$, $\tau$ an idempotent endomorphism of $A$, $(A, \tau)$ state-morphism algebra
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$$\theta_\tau = \{(x, y) \in A \times A : \tau(x) = \tau(y)\},$$
General Approach - State-Morphism Algebras

- $A$ an algebra of type $F$, $\tau$ an idempotent endomorphism of $A$, $(A, \tau)$ state-morphism algebra
- $\theta_\tau = \{(x, y) \in A \times A : \tau(x) = \tau(y)\}$,
- $\phi \subseteq A^2$, $\Phi(\phi)$, $\Phi_\tau(\phi)$ congruence generated by $\phi$ on $A$ and $(A, \tau)$
A an algebra of type $F$, $\tau$ an idempotent endomorphism of $A$, $(A, \tau)$ state-morphism algebra

$\theta_{\tau} = \{(x, y) \in A \times A : \tau(x) = \tau(y)\}$,

$\phi \subseteq A^2$, $\Phi(\phi)$, $\Phi_{\tau}(\phi)$ congruence generated by $\phi$ on $A$ and $(A, \tau)$

Lemma: For any $\phi \in \text{Con} \tau(A)$, we have $\theta_{\phi} \in \text{Con} (A, \tau)$, and $\theta_{\phi} \cap \tau(A)^2 = \phi$. In addition, $\theta_{\tau} \in \text{Con} (A, \tau)$, $\phi \subseteq \theta_{\phi}$, and $\Theta_{\tau}(\phi) \subseteq \theta_{\phi}$. 
Lemma: Let $\theta \in \text{Con } A$ be such that $\theta \subseteq \theta_{\tau}$. Then $\theta \in \text{Con } (A, \tau)$ holds.
• **Lemma:** Let $\theta \in \text{Con } A$ be such that $\theta \subseteq \theta_\tau$. Then $\theta \in \text{Con } (A, \tau)$ holds.

• **Lemma:** If $x, y \in \tau(A)$, then $\Theta(x, y) = \Theta_\tau(x, y)$. Consequently, $\Theta(\phi) = \Theta_\tau(\phi)$ whenever $\phi \subseteq \tau(A)^2$. 
• Lemma: Let $\theta \in \text{Con } A$ be such that $\theta \subseteq \theta_\tau$. Then $\theta \in \text{Con } (A, \tau)$ holds.

• Lemma: If $x, y \in \tau(A)$, then $\Theta(x, y) = \Theta_\tau(x, y)$. Consequently, $\Theta(\phi) = \Theta_\tau(\phi)$ whenever $\phi \subseteq \tau(A)^2$.

• if $(C, \tau) \hookrightarrow (B \times B, \tau_B)$, $(C, \tau)$ is said to be a subdiagonal state-morphism algebra.
Theorem 0.16 Let \((A, \tau)\) be a subdirectly irreducible state-morphism algebra such that \(A\) is subdirectly reducible. Then there is a subdirectly irreducible algebra \(B\) such that \((A, \tau)\) is \(B\)-subdiagonal.
• **Theorem 0.18** Let \((A, \tau)\) be a subdirectly irreducible state-morphism algebra such that \(A\) is subdirectly reducible. Then there is a subdirectly irreducible algebra \(B\) such that \((A, \tau)\) is \(B\)-subdiagonal.

• **Theorem 0.19** For every subdirectly irreducible state-morphism algebra \((A, \tau)\), there is a subdirectly irreducible algebra \(B\) such that \((A, \tau)\) is \(B\)-subdiagonal.
• **Theorem 0.20** Let \((A, \tau)\) be a subdirectly irreducible state-morphism algebra such that \(A\) is subdirectly reducible. Then there is a subdirectly irreducible algebra \(B\) such that \((A, \tau)\) is \(B\)-subdiagonal.

• **Theorem 0.21** For every subdirectly irreducible state-morphism algebra \((A, \tau)\), there is a subdirectly irreducible algebra \(B\) such that \((A, \tau)\) is \(B\)-subdiagonal.

• \(\mathcal{K}\) of algebras of the same type, \(I(\mathcal{K})\), \(H(\mathcal{K})\), \(S(\mathcal{K})\) and \(P(\mathcal{K})\) \(D(\mathcal{K})\)
Theorem 0.22  (1) For every class $\mathcal{K}$ of algebras of the same type $F$, 
$V(D(\mathcal{K})) = V(\mathcal{K})_\tau$.

(2) Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of same type algebras. Then $V(D(\mathcal{K}_1)) = V(D(\mathcal{K}_2))$ if and only if $V(\mathcal{K}_1) = V(\mathcal{K}_2)$. 
• **Theorem 0.24** (1) For every class $\mathcal{K}$ of algebras of the same type $F$, 
$V(D(\mathcal{K})) = V(\mathcal{K})_\tau$.
(2) Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of same type algebras. Then $V(D(\mathcal{K}_1)) = V(D(\mathcal{K}_2))$ if and only if $V(\mathcal{K}_1) = V(\mathcal{K}_2)$.

• **Theorem 0.25** If a system $\mathcal{K}$ of algebras of the same type $F$ generates the whole variety $V(F)$ of all algebras of type $F$, then the variety $V(F)_\tau$ of all state-morphism algebras $(A, \tau)$, where $A \in V(F)$, is generated by the class $\{D(A) : A \in \mathcal{K}\}$. 
Theorem 0.26  If $A$ is a subdirectly irreducible algebra, then any state-morphism algebra $(A, \tau)$ is subdirectly irreducible.
• **Theorem 0.28** If $A$ is a subdirectly irreducible algebra, then any state-morphism algebra $(A, \tau)$ is subdirectly irreducible.

• **Theorem 0.29** A variety $\mathcal{V}_\tau$ satisfy the CEP if and only if $\mathcal{V}$ satisfies the CEP.
Applications

- The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0, 1]_{MV})$. 
Applications

- The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0, 1]_{MV})$.
- The variety of all state-morphism BL-algebras is generated by the class $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}\}$.
Applications

• The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0, 1]_{MV})$.

• The variety of all state-morphism BL-algebras is generated by the class $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}\}$.

• The variety of all state-morphism MTL-algebras is generated by the class $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}_{lc}\}$.
Applications

- The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0, 1]_{MV})$.
- The variety of all state-morphism BL-algebras is generated by the class \( \{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T} \} \).
- The variety of all state-morphism MTL-algebras is generated by the class \( \{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}_{lc} \} \).
- The variety of all state-morphism naBL-algebras is generated by the class \( \{D(\mathbb{I}_t^{na}) : \mathbb{I}_t \in na\mathcal{T} \} \).
If a unital ℓ-group \((G, u)\) is double transitive, then \(D(\Gamma(G, u))\) generates the variety of state-morphism pseudo MV-algebras.
References


References


Thank you for your attention