State-morphism MV-algebras and Their Generalizations

Anatolij DVUREČENSKIJ

Mathematical Institute, Slovak Academy of Sciences,

Štefánikova 49, SK-814 73 Bratislava, Slovakia

E-mail: dvurecen@mat.savba.sk

The talk given at 11th INTERNATIONAL CONFERENCE ON FUZZY SET THEORY AND APPLICATIONS Liptovsk Jn, Slovak Republic, January 30 - February 3, 2012

ERDF OP R&D Projects CE meta-QUTE ITMS 26240120022.

States on MV-algebras

- M - MV-algebra, we define a partial operation $+$, via $a + b$ is defined iff $a \leq b^*$ iff $a \odot b = 0$, then $a + b := a \oplus b$.

States on MV-algebras

- M - MV-algebra, we define a partial operation $+$, via $a + b$ is defined iff $a \leq b^*$ iff $a \odot b = 0$, then $a + b := a \oplus b$.
- $+$ restriction of the ℓ-group addition

States on MV-algebras

- M - MV-algebra, we define a partial operation $+$, via $a + b$ is defined iff $a \leq b^*$ iff $a \odot b = 0$, then $a + b := a \oplus b$.
- $+$ restriction of the ℓ-group addition
- state- $s : M \to [0, 1]$, (i) $s(a + b) = s(a) + s(b)$, (ii) $s(1) = 1$.

States on MV-algebras

- M - MV-algebra, we define a partial operation $+$, via $a+b$ is defined iff $a \leq b^*$ iff $a \odot b = 0$, then $a+b := a \oplus b$.
- $+$ restriction of the ℓ-group addition
- state- $s : M \to [0,1]$, (i) $s(a+b) = s(a) + s(b)$, (ii) $s(1) = 1$.
- $\mathcal{S}(M)$ -set of states. $\mathcal{S}(M) \neq \emptyset$.

States on MV-algebras

- M - MV-algebra, we define a partial operation $+$, via $a+b$ is defined iff $a \leq b^*$ iff $a \odot b = 0$, then $a+b := a \oplus b$.
- $+$ restriction of the ℓ-group addition
- state- $s : M \to [0,1]$, (i) $s(a+b) = s(a) + s(b)$, (ii) $s(1) = 1$.
- $\mathcal{S}(M)$ -set of states. $\mathcal{S}(M) \neq \emptyset$.
- extremal state $s = \lambda s_1 + (1-\lambda)s_2$ for $\lambda \in (0,1) \Rightarrow s = s_1 = s_2$.

- $\{s_\alpha\} \to s$ iff $\lim_\alpha s_\alpha(a) \to s(a)$, $a \in M$.

- $\{s_\alpha\} \to s$ iff $\lim_\alpha s_\alpha(a) \to s(a)$, $a \in M$.
- $\mathcal{S}(E)$ - Hausdorff compact topological space, $\partial_e \mathcal{S}(M)$

- $\{s_\alpha\} \to s$ iff $\lim_\alpha s_\alpha(a) \to s(a)$, $a \in M$.
- $\mathcal{S}(E)$ - Hausdorff compact topological space, $\partial_e \mathcal{S}(M)$
- Krein-Mil'man $\mathcal{S}(M) = \mathrm{Cl}(\mathrm{ConHul}(\partial_e \mathcal{S}(M))$

- $\{s_\alpha\} \to s$ iff $\lim_\alpha s_\alpha(a) \to s(a)$, $a \in M$.
- $\mathcal{S}(E)$ - Hausdorff compact topological space, $\partial_e \mathcal{S}(M)$
- Krein-Mil'man $\mathcal{S}(M) = \mathrm{Cl}(\mathrm{ConHul}(\partial_e \mathcal{S}(M))$
- s is extremal iff $s(a \wedge b) = \min\{s(a), s(b)\}$ iff s is MV-homomorphism iff $\mathrm{Ker}(s)$ is a maximal ideal.

- $\{s_\alpha\} \to s$ iff $\lim_\alpha s_\alpha(a) \to s(a)$, $a \in M$.
- $\mathcal{S}(E)$ - Hausdorff compact topological space, $\partial_e \mathcal{S}(M)$
- Krein-Mil'man $\mathcal{S}(M) = \mathrm{Cl}(\mathrm{ConHul}(\partial_e \mathcal{S}(M))$
- s is extremal iff $s(a \wedge b) = \min\{s(a), s(b)\}$ iff s is MV-homomorphism iff $\mathrm{Ker}(s)$ is a maximal ideal.
- $s \leftrightarrow \mathrm{Ker}(s)$, 1-1 correspondence

- every maximal ideal is a kernel of a unique state

- every maximal ideal is a kernel of a unique state
- Kernel-hull topology = $\partial_e \mathcal{S}(E)$ set of extremal states

- every maximal ideal is a kernel of a unique state
- Kernel-hull topology = $\partial_e\mathcal{S}(E)$ set of extremal states
- Kroupa- Panti $a \mapsto \hat{a}$, $\hat{a}(s) := s(a)$,

$$s(a) = \int_{\partial_e\mathcal{S}(M)} \hat{a}(t)d\mu_s(t)$$

- every maximal ideal is a kernel of a unique state
- Kernel-hull topology = $\partial_e \mathcal{S}(E)$ set of extremal states
- Kroupa- Panti $a \mapsto \hat{a}$, $\hat{a}(s) := s(a)$,

$$s(a) = \int_{\partial_e \mathcal{S}(M)} \hat{a}(t) d\mu_s(t)$$

- μ_s - unique Borel regular σ-additive probability measure on $\mathcal{B}(\mathcal{S}(M))$ such that $\mu_s(\partial_e \mathcal{S}(M)) = 1$

State MV-algebras

- MV-algebras with a state are not universal algebras, and therefore, the do not provide an algebraizable logic for probability reasoning over many-valued events

State MV-algebras

- MV-algebras with a state are not universal algebras, and therefore, the do not provide an algebraizable logic for probability reasoning over many-valued events
- Flaminio-Montagna - introduce an algebraizable logic whose equivalent algebraic semantics is the variety of state MV-algebras

State MV-algebras

- MV-algebras with a state are not universal algebras, and therefore, the do not provide an algebraizable logic for probability reasoning over many-valued events
- Flaminio-Montagna - introduce an algebraizable logic whose equivalent algebraic semantics is the variety of state MV-algebras
- A state MV-algebra is a pair (M, τ), M - MV-algebra, τ unary operation on A s.t.

- $\tau(1) = 1$

- $\tau(1) = 1$
- $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$

- $\tau(1) = 1$
- $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$
- $\tau(x^*) = \tau(x)^*$

- $\tau(1) = 1$
- $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$
- $\tau(x^*) = \tau(x)^*$
- $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$

- $\tau(1) = 1$
- $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$
- $\tau(x^*) = \tau(x)^*$
- $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$
- τ -internal operator, state operator

Properties

- $\tau^2 = \tau$

Properties

- $\tau^2 = \tau$
- $\tau(M)$ is an MV-algebra and τ on $\tau(M)$ - identity

Properties

- $\tau^2 = \tau$
- $\tau(M)$ is an MV-algebra and τ on $\tau(M)$ - identity
- $\tau(x + y) = \tau(x) + \tau(y)$

Properties

- $\tau^2 = \tau$
- $\tau(M)$ is an MV-algebra and τ on $\tau(M)$ - identity
- $\tau(x + y) = \tau(x) + \tau(y)$
- $\tau(x \odot y) = \tau(x) \odot \tau(y)$ if $x \odot y = 0$.

Properties

- $\tau^2 = \tau$
- $\tau(M)$ is an MV-algebra and τ on $\tau(M)$ - identity
- $\tau(x + y) = \tau(x) + \tau(y)$
- $\tau(x \odot y) = \tau(x) \odot \tau(y)$ if $x \odot y = 0.$
- if (M, τ) is s.i., then $\tau(M)$ is a chain

Properties

- $\tau^2 = \tau$
- $\tau(M)$ is an MV-algebra and τ on $\tau(M)$ - identity
- $\tau(x + y) = \tau(x) + \tau(y)$
- $\tau(x \odot y) = \tau(x) \odot \tau(y)$ if $x \odot y = 0.$
- if (M, τ) is s.i., then $\tau(M)$ is a chain
- if (M, τ) is s.i., then M is not necessarily a chain

- F -filter, τ-filter if $\tau(F) \subseteq F$.

- F -filter, τ-filter if $\tau(F) \subseteq F$.
- 1-1 correspondence congruences and τ-filters

- F -filter, τ-filter if $\tau(F) \subseteq F$.
- 1-1 correspondence congruences and τ-filters
- $M = [0,1] \times [0,1]$, $\tau(x,y) = (x,x)$ s.i. - not chain

- F -filter, τ-filter if $\tau(F) \subseteq F$.
- 1-1 correspondence congruences and τ-filters
- $M = [0,1] \times [0,1]$, $\tau(x,y) = (x,x)$ s.i. - not chain
- state-morphism (M,τ), τ is an idempotent endomorphism

- F -filter, τ-filter if $\tau(F) \subseteq F$.
- 1-1 correspondence congruences and τ-filters
- $M = [0,1] \times [0,1]$, $\tau(x,y) = (x,x)$ s.i. - not chain
- state-morphism (M,τ), τ is an idempotent endomorphism
- s state on M, $[0,1] \otimes M$, $\tau_s(\alpha \otimes a) := \alpha \cdot s(a) \otimes 1$

- $([0,1]\otimes, \tau_s)$ is an SMV-algebra.

- $([0,1]\otimes, \tau_s)$ is an SMV-algebra.
- $([0,1]\otimes, \tau_s)$ is an SMMV-algebra iff s is an extremal state

- $([0,1]\otimes, \tau_s)$ is an SMV-algebra.
- $([0,1]\otimes, \tau_s)$ is an SMMV-algebra iff s is an extremal state
- if M is a chain, every SMV-algebra (M, τ) is an SMMV-algebra

- $([0,1]\otimes, \tau_s)$ is an SMV-algebra.
- $([0,1]\otimes, \tau_s)$ is an SMMV-algebra iff s is an extremal state
- if M is a chain, every SMV-algebra (M, τ) is an SMMV-algebra
- if $\tau(M) \in \mathsf{V}(S_1, \ldots, S_n)$ for some $n \geq 1$, then (M, τ) is an SMMV-algebra

- $([0,1]_\otimes, \tau_s)$ is an SMV-algebra.
- $([0,1]_\otimes, \tau_s)$ is an SMMV-algebra iff s is an extremal state
- if M is a chain, every SMV-algebra (M, τ) is an SMMV-algebra
- if $\tau(M) \in \mathsf{V}(S_1, \ldots, S_n)$ for some $n \geq 1$, then (M, τ) is an SMMV-algebra
- Iff $\tau((n+1)x) = \tau(nx)$

Other examples

- Let (G, u) be a unital Riesz space and let $A = \Gamma(G, u)$. Choose an endomorphism $s : A \to A$ such that $s \circ s = s$ and fix a real number $\lambda \in [0, 1]$. Define a mapping $s_\lambda : A^2 \to A$ by

$$s_\lambda(x, y) = \lambda s(x) + (1 - \lambda)s(y), \quad (x, y) \in A^2,$$

and set $\tau_{\lambda,s} : A^3 \to A^3$ by

$$\tau_{\lambda,s}(x, y, z) = (s_\lambda(x, y), s_\lambda(x, y), s_\lambda(x, y)), \quad (x, y, z) \in A$$

- Then $D(A)_{\lambda,s} := (A^3, \tau_{\lambda,s})$ is a state MV-algebra such that it is a state morphism MV-algebra iff $\lambda \in \{0,1\}$.

$$\mathrm{Ker}(\tau_{\lambda,s}) = \begin{cases} \mathrm{Ker}(s) \times \mathrm{Ker}(s) \times A & \text{if } \lambda \in (0,1), \\ A \times \mathrm{Ker}(s) \times A & \text{if } \lambda = 0, \\ \mathrm{Ker}(s) \times A \times A & \text{if } \lambda = 1. \end{cases}$$

Hence, if A is a subdirectly irreducible MV-algebra and s is faithful, then neither $(A, \tau_{0,s})$ nor $(A, \tau_{1,s})$ is a subdirectly irreducible state morphism MV-algebra.

- whilst if $\lambda \in (0,1)$, $D(A_{\lambda,s})$ is a subdirectly irreducible state MV-algebra.

- whilst if $\lambda \in (0,1)$, $D(A_{\lambda,s})$ is a subdirectly irreducible state MV-algebra.
- Let $A = [0,1]$ be the standard MV-algebra and let s be the identity on A. If $\lambda \in \{0,1\}$, then $D([0,1]_{1,s})$ generates the variety of state morphism MV-algebras

- whilst if $\lambda \in (0,1)$, $D(A_{\lambda,s})$ is a subdirectly irreducible state MV-algebra.
- Let $A = [0,1]$ be the standard MV-algebra and let s be the identity on A. If $\lambda \in \{0,1\}$, then $D([0,1]_{1,s})$ generates the variety of state morphism MV-algebras
- Does $D([0,1]_{\lambda,s})$ for a fixed $\lambda \in (0,1)$ and the identity s on $[0,1]$ generate the variety of state MV-algebras ?

State BL-algebras

- M - BL-algebra. A map $\tau : M \to M$ s.t.

$$(1)_{BL}\ \tau(0) = 0;$$
$$(2)_{BL}\ \tau(x \to y) = \tau(x) \to \tau(x \wedge y);$$
$$(3)_{BL}\ \tau(x \odot y) = \tau(x) \odot \tau(x \to (x \odot y));$$
$$(4)_{BL}\ \tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y);$$
$$(5)_{BL}\ \tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$$

state-operator on M, pair (M, τ) - *state BL-algebra*

State BL-algebras

- M - BL-algebra. A map $\tau : M \to M$ s.t.

$(1)_{BL}$ $\tau(0) = 0$;

$(2)_{BL}$ $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$;

$(3)_{BL}$ $\tau(x \odot y) = \tau(x) \odot \tau(x \to (x \odot y))$;

$(4)_{BL}$ $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y)$;

$(5)_{BL}$ $\tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$

state-operator on M, pair (M, τ) - *state BL-algebra*

- If $\tau : M \to M$ is a BL-endomorphism s.t. $\tau \circ \tau = \tau$, - *state-morphism operator* and the couple (M, τ) - *state-morphism BL-algebra*.

- every state operator on a linear BL-algebra is a state-morphism

- every state operator on a linear BL-algebra is a state-morphism
- **Example 0.2** Let M be a BL-algebra. On $M \times M$ we define two operators, τ_1 and τ_2, as follows

$$\tau_1(a,b) = (a,a), \quad \tau_2(a,b) = (b,b), \quad (a,b) \in M \times M. \tag{2.0}$$

Then τ_1 and τ_2 are two state-morphism operators on $M \times M$.

- every state operator on a linear BL-algebra is a state-morphism
- **Example 0.3** Let M be a BL-algebra. On $M \times M$ we define two operators, τ_1 and τ_2, as follows

$$\tau_1(a,b) = (a,a), \quad \tau_2(a,b) = (b,b), \quad (a,b) \in M \times M. \tag{2.0}$$

  Then τ_1 and τ_2 are two state-morphism operators on $M \times M$.
- Ker$(\tau) = \{a \in M : \tau(a) = 1\}$.

- We say that two subhoops, A and B, of a BL-algebra M have the *disjunction property* if for all $x \in A$ and $y \in B$, if $x \vee y = 1$, then either $x = 1$ or $y = 1$.

- We say that two subhoops, A and B, of a BL-algebra M have the *disjunction property* if for all $x \in A$ and $y \in B$, if $x \vee y = 1$, then either $x = 1$ or $y = 1$.
- **Lemma 0.5** *Suppose that (M, τ) is a state BL-algebra. Then:*

(1) *If τ is faithful, then (M, τ) is a subdirectly irreducible state BL-algebra if and only if $\tau(M)$ is a subdirectly irreducible BL-algebra.*

Now let (M, τ) be subdirectly irreducible. Then:

- (2) Ker(τ) is (either trivial or) a subdirectly irreducible hoop.

  (3) Ker(τ) and $\tau(M)$ have the disjunction property.

- (2) Ker(τ) is (either trivial or) a subdirectly irreducible hoop.

  (3) Ker(τ) and $\tau(M)$ have the disjunction property.

- **Theorem 0.7** *Let (M, τ) be a state BL-algebra satisfying conditions* (1), (2) *and* (3) *in the last Lemma. Then (M, τ) is subdirectly irreducible.*

- **Theorem 0.8** *A state-morphism BL-algebra* (M, τ) *is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*

- **Theorem 0.9** *A state-morphism BL-algebra* (M, τ) *is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.

- **Theorem 0.10** *A state-morphism BL-algebra (M, τ) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.
- (ii) The state-morphism operator τ is not faithful, M has no nontrivial Boolean elements, and the BL-reduct M of (M, τ) is a local BL-algebra, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\mathrm{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.

- **Theorem 0.11** *A state-morphism BL-algebra (M, τ) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.
- (ii) The state-morphism operator τ is not faithful, M has no nontrivial Boolean elements, and the BL-reduct M of (M, τ) is a local BL-algebra, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\mathrm{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.

- **Theorem 0.12** *A state-morphism BL-algebra (M, τ) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.
- (ii) The state-morphism operator τ is not faithful, M has no nontrivial Boolean elements, and the BL-reduct M of (M, τ) is a local BL-algebra, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\mathrm{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.

- **Theorem 0.13** *A state-morphism BL-algebra (M, τ) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.
- (ii) The state-morphism operator τ is not faithful, M has no nontrivial Boolean elements, and the BL-reduct M of (M, τ) is a local BL-algebra, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\mathrm{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.

- Moreover, M is linearly ordered if and only if $\mathrm{Rad}_1(M)$ is linearly ordered, and in such a case, M is a subdirectly irreducible BL-algebra such that if F is the smallest nontrivial state-filter for (M, τ), then F is the smallest nontrivial BL-filter for M.

- Moreover, M is linearly ordered if and only if $\text{Rad}_1(M)$ is linearly ordered, and in such a case, M is a subdirectly irreducible BL-algebra such that if F is the smallest nontrivial state-filter for (M, τ), then F is the smallest nontrivial BL-filter for M.
- If $\text{Rad}(M) = \text{Ker}(\tau)$, then M is linearly ordered.

- (iii) The state-morphism operator τ is not faithful, M has a nontrivial Boolean element. There are a linearly ordered BL-algebra A, a subdirectly irreducible BL-algebra B, and an injective BL-homomorphism $h: A \to B$ such that (M, τ) is isomorphic as a state-morphism BL-algebra with the state-morphism BL-algebra $(A \times B, \tau_h)$, where $\tau_h(x, y) = (x, h(x))$ for any $(x, y) \in A \times B$.

Varieties of SMMV-algebras

- Komori - countably many subvarieties of MV-algebras

Varieties of SMMV-algebras

- Komori - countably many subvarieties of MV-algebras
- $\mathcal{V}$-variety of MV-algebras, $\mathcal{V}_\tau$ -system of SMMV-algebras (M, τ) s.t $M \in \mathcal{V} \in \mathcal{V}$.

Varieties of SMMV-algebras

- Komori - countably many subvarieties of MV-algebras
- $\mathcal{V}$-variety of MV-algebras, $\mathcal{V}_\tau$ -system of SMMV-algebras (M, τ) s.t $M \in \mathcal{V} \in \mathcal{V}$.
- $D(M) := (M \times M, \tau_M)$

Varieties of SMMV-algebras

- Komori - countably many subvarieties of MV-algebras
- $\mathcal{V}$-variety of MV-algebras, $\mathcal{V}_\tau$ -system of SMMV-algebras (M, τ) s.t $M \in \mathcal{V} \in \mathcal{V}$.
- $D(M) := (M \times M, \tau_M)$
- $\mathsf{V}(D) = \mathsf{V}(M)_\tau$

Varieties of SMMV-algebras

- Komori - countably many subvarieties of MV-algebras
- $\mathcal{V}$-variety of MV-algebras, $\mathcal{V}_\tau$ -system of SMMV-algebras (M, τ) s.t $M \in \mathcal{V} \in \mathcal{V}$.
- $D(M) := (M \times M, \tau_M)$
- $\mathsf{V}(D) = \mathsf{V}(M)_\tau$
- $\mathcal{SMMV} = \mathsf{V}(D([0,1]))$

Varieties of SMMV-algebras

- Komori - countably many subvarieties of MV-algebras
- $\mathcal{V}$-variety of MV-algebras, $\mathcal{V}_\tau$ -system of SMMV-algebras (M,τ) s.t $M \in \mathcal{V} \in \mathcal{V}$.
- $D(M) := (M \times M, \tau_M)$
- $\mathsf{V}(D) = \mathsf{V}(M)_\tau$
- $\mathcal{SMMV} = \mathsf{V}(D([0,1]))$
- $\mathcal{P}_\tau = \mathsf{V}(D(C))$, $\mathcal{P}$ perfect MV-algebras, C-Chang

- Theorem: $\mathcal{VI} \subseteq \mathcal{VR} \subseteq \mathcal{VL} \subseteq \mathcal{V}_{\mathcal{T}}$. and all inclusions are proper of $\mathcal{V}$ is not finitely generated.

- Theorem: $\mathcal{VI} \subseteq \mathcal{VR} \subseteq \mathcal{VL} \subseteq \mathcal{V}_\tau$. and all inclusions are proper of $\mathcal{V}$ is not finitely generated.
- Theorem: Representable SMMV-algebras:

$$\tau(x) \vee (x \to (\tau(y) \leftrightarrow y)) = 1.$$

- Theorem: $\mathcal{VI} \subseteq \mathcal{VR} \subseteq \mathcal{VL} \subseteq \mathcal{V}_\tau$. and all inclusions are proper of $\mathcal{V}$ is not finitely generated.
- Theorem: Representable SMMV-algebras:

$$\tau(x) \vee (x \to (\tau(y) \leftrightarrow y)) = 1.$$

- also for BL-algebra

- Theorem: $\mathcal{VI} \subseteq \mathcal{VR} \subseteq \mathcal{VL} \subseteq \mathcal{V}_\tau$. and all inclusions are proper of $\mathcal{V}$ is not finitely generated.
- Theorem: Representable SMMV-algebras:

$$\tau(x) \vee (x \to (\tau(y) \leftrightarrow y)) = 1.$$

- also for BL-algebra
- Theorem: $\mathcal{VL}$ - generated by those (M, τ), M is local

- Theorem: $\mathcal{VI} \subseteq \mathcal{VR} \subseteq \mathcal{VL} \subseteq \mathcal{V}_\tau$. and all inclusions are proper of $\mathcal{V}$ is not finitely generated.
- Theorem: Representable SMMV-algebras:

$$\tau(x) \vee (x \to (\tau(y) \leftrightarrow y)) = 1.$$

- also for BL-algebra
- Theorem: $\mathcal{VL}$ - generated by those (M, τ), M is local
-

$$(\tau(x) \leftrightarrow x)^* \leq (\tau(x) \leftrightarrow x).$$

Uncountable many subvarieties

- $[0,1]^*$ ultrapower, for positive infinitesimal $\epsilon \in [0,1]^*$

Uncountable many subvarieties

- $[0,1]^*$ ultrapower, for positive infinitesimal $\epsilon \in [0,1]^*$
- X subset of prime numbers, $A(X)$ MV-algebra generated by ϵ and $\frac{n}{m}$ s.t

Uncountable many subvarieties

- $[0,1]^*$ ultrapower, for positive infinitesimal $\epsilon \in [0,1]^*$
- X subset of prime numbers, $A(X)$ MV-algebra generated by ϵ and $\frac{n}{m}$ s.t
- (1) either $n = 0$ or g.c.d$(n, m) = 1$

Uncountable many subvarieties

- $[0,1]^*$ ultrapower, for positive infinitesimal $\epsilon \in [0,1]^*$
- X subset of prime numbers, $A(X)$ MV-algebra generated by ϵ and $\frac{n}{m}$ s.t
- (1) either $n = 0$ or g.c.d$(n, m) = 1$
- $\forall p \in X$, p does not divide m

Uncountable many subvarieties

- $[0,1]^*$ ultrapower, for positive infinitesimal $\epsilon \in [0,1]^*$
- X subset of prime numbers, $A(X)$ MV-algebra generated by ϵ and $\frac{n}{m}$ s.t
- (1) either $n = 0$ or g.c.d$(n, m) = 1$
- $\forall p \in X$, p does not divide m
- $\tau(x)$= standard part of x

Uncountable many subvarieties

- $[0,1]^*$ ultrapower, for positive infinitesimal $\epsilon \in [0,1]^*$
- X subset of prime numbers, $A(X)$ MV-algebra generated by ϵ and $\frac{n}{m}$ s.t
- (1) either $n = 0$ or g.c.d$(n, m) = 1$
- $\forall p \in X$, p does not divide m
- $\tau(x)$= standard part of x
- $(A(X), \tau)$ is linearly ordered SMMV-algebra

- if $X \neq Y$, then $\mathsf{V}(A(X)) \neq \mathsf{V}(A(Y))$

- if $X \neq Y$, then $\mathsf{V}(A(X)) \neq \mathsf{V}(A(Y))$
- Theorem: Between $\mathcal{MVI}$ and $\mathcal{MVR}$ there is uncountably many varieties

Generators of SMBL-algebras

- *t-norm*- function $t : [0,1] \times [0,1] \to [0,1]$ such that (i) t is commutative, associative, (ii) $t(x,1) = x$, $x \in [0,1]$, and (iii) t is nondecreasing.

Generators of SMBL-algebras

- *t-norm*- function $t : [0,1] \times [0,1] \to [0,1]$ such that (i) t is commutative, associative, (ii) $t(x,1) = x$, $x \in [0,1]$, and (iii) t is nondecreasing.
- If t is continuous, we define $x \odot_t y = t(x,y)$ and $x \to_t y = \sup\{z \in [0,1] : t(z,x) \leq y\}$ for $x, y \in [0,1]$, then $\mathbb{I}_t := ([0,1], \min, \max, \odot_t, \to_t, 0, 1)$ is a BL-algebra.

Generators of SMBL-algebras

- *t-norm*- function $t : [0,1] \times [0,1] \to [0,1]$ such that (i) t is commutative, associative, (ii) $t(x,1) = x$, $x \in [0,1]$, and (iii) t is nondecreasing.
- If t is continuous, we define $x \odot_t y = t(x,y)$ and $x \to_t y = \sup\{z \in [0,1] : t(z,x) \leq y\}$ for $x, y \in [0,1]$, then $\mathbb{I}_t := ([0,1], \min, \max, \odot_t, \to_t, 0, 1)$ is a BL-algebra.
- The variety of all BL-algebras is generated by all $\mathbb{I}_t$ with a continuous t-norm t.

- $\mathcal{T}$ denotes the system of all BL-algebras $D(\mathbb{I}_t)$, where t is a continuous t-norm on the interval $[0,1]$,

- $\mathcal{T}$ denotes the system of all BL-algebras $D(\mathbb{I}_t)$, where t is a continuous t-norm on the interval $[0,1]$,
- **Theorem 0.15** *The variety of all state-morphism BL-algebras is generated by the class* $\mathcal{T}$.

General Approach - State-Morphism Alge

- A an algebra of type F, τ an idempotent endomorphism of A, (A, τ) state-morphism algebra

General Approach - State-Morphism Alge

- A an algebra of type F, τ an idempotent endomorphism of A, (A, τ) state-morphism algebra
- $\theta_\tau = \{(x, y) \in A \times A : \tau(x) = \tau(y)\}$,

General Approach - State-Morphism Alge

- A an algebra of type F, τ an idempotent endomorphism of A, (A, τ) state-morphism algebra
- $\theta_\tau = \{(x, y) \in A \times A : \tau(x) = \tau(y)\}$,
- $\phi \subseteq A^2$, $\Phi(\phi)$, $\Phi_\tau(\phi)$ congruence generated by ϕ on A and (A, τ)

General Approach - State-Morphism Alge

- A an algebra of type F, τ an idempotent endomorphism of A, (A, τ) state-morphism algebra
- $\theta_\tau = \{(x, y) \in A \times A : \tau(x) = \tau(y)\}$,
- $\phi \subseteq A^2$, $\Phi(\phi)$, $\Phi_\tau(\phi)$ congruence generated by ϕ on A and (A, τ)
- Lemma: For any $\phi \in \mathrm{Con}\,\tau(\mathbf{A})$, we have $\theta_\phi \in \mathrm{Con}\,(\mathbf{A}, \tau)$, and $\theta_\phi \cap \tau(A)^2 = \phi$. In addition, $\theta_\tau \in \mathrm{Con}\,(\mathbf{A}, \tau)$, $\phi \subseteq \theta_\phi$, and $\Theta_\tau(\phi) \subseteq \theta_\phi$.

- Lemma: Let $\theta \in \mathrm{Con}\,\mathbf{A}$ be such that $\theta \subseteq \theta_\tau$. Then $\theta \in \mathrm{Con}\,(\mathbf{A}, \tau)$ holds.

- Lemma: Let $\theta \in \mathrm{Con}\,\mathbf{A}$ be such that $\theta \subseteq \theta_\tau$. Then $\theta \in \mathrm{Con}\,(\mathbf{A}, \tau)$ holds.
- Lemma: If $x, y \in \tau(\mathbf{A})$, then $\Theta(x,y) = \Theta_\tau(x,y)$. Consequently, $\Theta(\phi) = \Theta_\tau(\phi)$ whenever $\phi \subseteq \tau(A)^2$.

- Lemma: Let $\theta \in \mathrm{Con}\,\mathbf{A}$ be such that $\theta \subseteq \theta_\tau$. Then $\theta \in \mathrm{Con}\,(\mathbf{A}, \tau)$ holds.
- Lemma: If $x, y \in \tau(\mathbf{A})$, then $\Theta(x, y) = \Theta_\tau(x, y)$. Consequently, $\Theta(\phi) = \Theta_\tau(\phi)$ whenever $\phi \subseteq \tau(A)^2$.
- if $(\mathbf{C}, \tau) \hookrightarrow (\mathbf{B} \times \mathbf{B}, \tau_B)$, $(\mathbf{C}, \tau)$ is said to be a *subdiagonal* state-morphism algebra

- **Theorem 0.16** *Let* (A, τ) *be a subdirectly irreducible state-morphism algebra such that* A *is subdirectly reducible. Then there is a subdirectly irreducible algebra* B *such that* (A, τ) *is* B*-subdiagonal.*

- **Theorem 0.18** *Let* (A, τ) *be a subdirectly irreducible state-morphism algebra such that* A *is subdirectly reducible. Then there is a subdirectly irreducible algebra* B *such that* (A, τ) *is* B*-subdiagonal.*
- **Theorem 0.19** *For every subdirectly irreducible state-morphism algebra* (A, τ)*, there is a subdirectly irreducible algebra* B *such that* (A, τ) *is* B*-subdiagonal.*

- **Theorem 0.20** *Let $(\mathbf{A}, \tau)$ be a subdirectly irreducible state-morphism algebra such that $\mathbf{A}$ is subdirectly reducible. Then there is a subdirectly irreducible algebra $\mathbf{B}$ such that $(\mathbf{A}, \tau)$ is $\mathbf{B}$-subdiagonal.*
- **Theorem 0.21** *For every subdirectly irreducible state-morphism algebra $(\mathbf{A}, \tau)$, there is a subdirectly irreducible algebra $\mathbf{B}$ such that $(\mathbf{A}, \tau)$ is $\mathbf{B}$-subdiagonal.*
- $\mathcal{K}$ of algebras of the same type, I$(\mathcal{K})$, H$(\mathcal{K})$, S$(\mathcal{K})$ and P$(\mathcal{K})$ D$(\mathcal{K})$

- **Theorem 0.22** (1) *For every class* $\mathcal{K}$ *of algebras of the same type* F, $\mathsf{V}(\mathsf{D}(\mathcal{K})) = \mathsf{V}(\mathcal{K})_\tau$.
(2) *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be two classes of same type algebras. Then* $\mathsf{V}(D(\mathcal{K}_1)) = \mathsf{V}(D(\mathcal{K}_2))$ *if and only if* $\mathsf{V}(\mathcal{K}_1) = \mathsf{V}(\mathcal{K}_2)$.

- **Theorem 0.24** (1) *For every class* $\mathcal{K}$ *of algebras of the same type* F, $\mathsf{V}(\mathsf{D}(\mathcal{K})) = \mathsf{V}(\mathcal{K})_\tau$.
(2) *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be two classes of same type algebras. Then* $\mathsf{V}(D(\mathcal{K}_1)) = \mathsf{V}(D(\mathcal{K}_2))$ *if and only if* $\mathsf{V}(\mathcal{K}_1) = \mathsf{V}(\mathcal{K}_2)$.
- **Theorem 0.25** *If a system* $\mathcal{K}$ *of algebras of the same type* F *generates the whole variety* $\mathcal{V}(F)$ *of all algebras of type* F, *then the variety* $\mathcal{V}(F)_\tau$ *of all state-morphism algebras* $(\mathbf{A}, \tau)$, *where* $\mathbf{A} \in \mathcal{V}(F)$, *is generated by the class* $\{D(\mathbf{A}) : \mathbf{A} \in \mathcal{K}\}$.

- **Theorem 0.26** *If* A *is a subdirectly irreducible algebra, then any state-morphism algebra* (A, τ) *is subdirectly irreducible.*

- **Theorem 0.28** *If* $\mathbf{A}$ *is a subdirectly irreducible algebra, then any state-morphism algebra* $(\mathbf{A}, \tau)$ *is subdirectly irreducible.*
- **Theorem 0.29** *A variety* $\mathcal{V}_\tau$ *satisfy the CEP if and only if* $\mathcal{V}$ *satisfies the CEP.*

Applications

- The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0,1]_{MV})$.

Applications

- The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0,1]_{MV})$.
- The variety of all state-morphism BL-algebras is generated by the class $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}\}$.

Applications

- The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0,1]_{MV})$.
- The variety of all state-morphism BL-algebras is generated by the class $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}\}$.
- The variety of all state-morphism MTL-algebras is generated by the class $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}_{lc}\}$.

Applications

- The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0,1]_{MV})$.
- The variety of all state-morphism BL-algebras is generated by the class $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}\}$.
- The variety of all state-morphism MTL-algebras is generated by the class $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}_{lc}\}$.
- The variety of all state-morphism naBL-algebras is generated by the class $\{D(\mathbb{I}_t^{na}) : \mathbb{I}_t \in na\mathcal{T}\}$.

- If a unital ℓ-group (G, u) is double transitive, then $D(\Gamma(G, u))$ generates the variety of state-morphism pseudo MV-algebras.

References

- A. Di Nola, A. Dvurečenskij, *State-morphism MV-algebras,* Ann. Pure Appl. Logic **161** (2009), 161–173.

References

- A. Di Nola, A. Dvurečenskij, *State-morphism MV-algebras,* Ann. Pure Appl. Logic **161** (2009), 161–173.
- A. Di Nola, A. Dvurečenskij, A. Lettieri, *Erratum “State-morphism MV-algebras” [Ann. Pure Appl. Logic 161 (2009) 161-173],* Ann. Pure Appl. Logic **161** (2010), 1605–1607.

References

- A. Di Nola, A. Dvurečenskij, *State-morphism MV-algebras,* Ann. Pure Appl. Logic **161** (2009), 161–173.
- A. Di Nola, A. Dvurečenskij, A. Lettieri, *Erratum "State-morphism MV-algebras" [Ann. Pure Appl. Logic 161 (2009) 161-173],* Ann. Pure Appl. Logic **161** (2010), 1605–1607.
- A. Dvurečenskij, *Subdirectly irreducible state-morphism BL-algebras,* Archive Math. Logic **50** (2011), 145–160.

- A. Dvurečenskij, T. Kowalski, F. Montagna, *State morphism MV-algebras,* Inter. J. Approx. Reasoning **52** (2011), 1215–1228. http://arxiv.org/abs/1102.1088

- A. Dvurečenskij, T. Kowalski, F. Montagna, *State morphism MV-algebras,* Inter. J. Approx. Reasoning **52** (2011), 1215–1228. http://arxiv.org/abs/1102.1088
- M. Botur, A. Dvurečenskij, T. Kowalski, *On normal-valued basic pseudo hoops,* Soft Computing DOI: 10.1007/s00500-011-0763-7

- A. Dvurečenskij, T. Kowalski, F. Montagna, *State morphism MV-algebras,* Inter. J. Approx. Reasoning **52** (2011), 1215–1228. http://arxiv.org/abs/1102.1088
- M. Botur, A. Dvurečenskij, T. Kowalski, *On normal-valued basic pseudo hoops,* Soft Computing DOI: 10.1007/s00500-011-0763-7
- A. Di Nola, A. Dvurečenskij, A. Lettieri, *On varieties of MV-algebras with internal states,* Inter. J. Approx. Reasoning **51** (2010), 680–694.

- A. Dvurečenskij, T. Kowalski, F. Montagna, *State morphism MV-algebras,* Inter. J. Approx. Reasoning **52** (2011), 1215–1228. http://arxiv.org/abs/1102.1088
- M. Botur, A. Dvurečenskij, T. Kowalski, *On normal-valued basic pseudo hoops,* Soft Computing DOI: 10.1007/s00500-011-0763-7
- A. Di Nola, A. Dvurečenskij, A. Lettieri, *On varieties of MV-algebras with internal states,* Inter. J. Approx. Reasoning **51** (2010), 680–694.
- L.C. Ciungu, A. Dvurečenskij, M. Hyčko, *State BL-algebras,* Soft Computing

- M. Botur, A. Dvurečenskij, *State-morphism algebras - general approach,* http://arxiv.org/submit/230594

Thank you for your attention