On aggregation of graded properties of fuzzy relations

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Introduction

Definition 1 (Zadeh 1965)**.** A fuzzy relation in $X \neq \emptyset$ is an arbitrary function $R : X \times X \to [0,1]$. The family of all fuzzy relations in X is denoted by FR(X).

Definition 2 (Fodor, Roubens 1994)**.** Let $F : [0,1]^n \to [0,1]$, $R_1, ..., R_n \in FR(X)$. By aggregation fuzzy relation we call $R \in FR(X)$,

$$R(x,y) = F(R_1(x,y), ..., R_n(x,y)), \quad x, y \in X.$$

An aggregation function F preserves a property of fuzzy relations if for every $R_1, ..., R_n \in FR(X)$ having this property, R also has this property.

Definition 3 (Calvo et al. 2002)**.** Let $n \geqslant 2$. $F : [0,1]^n \to [0,1]$ is called an aggregation function, if it is increasing with respect to any variable and fulfils

$$F(0, ..., 0) = 0, \quad F(1, ..., 1) = 1.$$

Definition 4 (Klement et al. 2000)**.** Triangular norm $T : [0,1]^2 \to [0,1]$ (triangular conorm $S : [0,1]^2 \to [0,1]$) is an arbitrary associative, commutative, increasing in both variables operation having a neutral element $e = 1$ ($e = 0$).

Example 1. Let $\varphi : [0,1] \to \mathbb{R}$ be continuous, strictly monotonic function. A quasi–arithmetic mean is the function

$$F(t_1, ..., t_n) = \varphi^{-1}(\frac{1}{n}\sum_{k=1}^{n}\varphi(t_k)),\ t_1, ..., t_n \in [0,1],$$

Median is the function

$$\mathrm{med}(t_1, \dots, t_n) = \begin{cases} \frac{s_k + s_{k+1}}{2}, & \text{if } n = 2k \\ s_{k+1}, & \text{if } n = 2k+1 \end{cases},\ t_1, ..., t_n \in [0,1],$$

where $(s_1, \dots, s_n)$ is the increasing permutation of the sequence $(t_1, \dots, t_n)$, so $s_1 \leqslant \dots \leqslant s_n$.

Example 2. Projections $P_k(t_1, \dots, t_n) = t_k$, $k \in \{1, ..., n\}$ preserve each property of fuzzy relations because for $F = P_k$ we get $R_F = R_k$.

Remark 1. If *card* $X = n$, $X = \{x_1, ..., x_n\}$, then $R \in FR(X)$ may be presented by a matrix $R = [r_{ik}]$, where $r_{ik} = R(x_i, x_k)$, $i, k = 1, ..., n$.

Motivation

Multicriteria decision making

Let $card\ X = m$, $m \in \mathbb{N}$, $X = \{x_1, ..., x_m\}$ – a set of alternatives
A decision maker has to
— choose among alternatives („choice problem")
— rank („ranking problem")
$K = \{k_1, ..., k_n\}$ – a set of criteria on the base of which the alternatives are evaluated.
$R_1, ..., R_n$ – fuzzy relations corresponding to each criterion represented by matrices, where $R_k : X \times X \to [0, 1]$, $k = 1, ..., n$, $n \in \mathbb{N}$, $R_k(x_i, x_j) = r^k_{ij}$, $1 \leqslant i, j \leqslant m$.
For example
r^k_{ij} – an intensity with which x_i is better than x_j under $k \in K$,
$r^k_{ij} = 1$ – „x_i is absolutely better than x_j under criterion k",
$r^k_{ij} = 0$ – „x_j is absolutely better than x_i under criterion k",
$r^k_{ij} = 0.5$ – „x_i is equally good as x_j under criterion k".

Relation $R = F(R_1, ..., R_n)$ is supposed to help a decision maker to make up their mind.

Reflexivity

Definition 5 (Drewniak 1989)**.** Let $\alpha \in [0,1]$. $R \in FR(X)$ is α–reflexive, if

$$\underset{x \in X}{\forall} \; R(x,x) \geqslant \alpha.$$

Theorem 1. *Let $\alpha \in [0,1]$. $F : [0,1]^n \to [0,1]$ preserves α–reflexivity of fuzzy relations, iff*

$$F|_{[\alpha,1]^n} \geqslant \alpha.$$

Theorem 2. *$F : [0,1]^n \to [0,1]$ preserves α–reflexivity of fuzzy relations for arbitrary $\alpha \in [0,1]$ iff $F \geqslant \min$.*

Corollary 1. *Every quasi–arithmetic mean preserves α–reflexivity of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Theorem 3. *Let $\alpha_1, ..., \alpha_n \in [0,1]$, $F : [0,1]^n \to [0,1]$ be increasing in each variable. If relations $R_i \in FR(X)$ are α_i–reflexive for $i = 1, ..., n$, then relation $R = F(R_1, ..., R_n)$ is α–reflexive, for $\alpha = F(\alpha_1, ..., \alpha_n)$.*

Theorem 4. *Let $\alpha \in [0,1]$, $F \leqslant \min$. If $R = F(R_1, \ldots, R_n)$ is α-reflexive, then all relations $R_1, ..., R_n$ are α-reflexive.*

Example 3. Let $card\, X = 2$. We consider fuzzy relations with matrices:

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$T_1 = \max(R, S) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, T_2 = \frac{R+S}{2} = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

Relation T_1 is α-reflexive for $\alpha \in [0,1]$, T_2 for $\alpha \in [0, 0.5]$, but relations R, S do not have this property for any $\alpha \in [0,1]$.

Irreflexivity

Definition 6 (Drewniak 1989)**.** Let $\alpha \in [0,1]$. $R \in FR(X)$ is α–irreflexive, if

$$\forall_{x \in X} R(x,x) \leqslant 1-\alpha.$$

Theorem 5. *Let $\alpha \in [0,1]$. $F : [0,1]^n \to [0,1]$ preserves α–irreflexivity of fuzzy relations iff*

$$F|_{[0,1-\alpha]^n} \leqslant 1-\alpha.$$

Theorem 6. *$F : [0,1]^n \to [0,1]$ preserves α–irreflexivity of fuzzy relations for arbitrary $\alpha \in [0,1]$ iff $F \leqslant \max$.*

Corollary 2. *Every quasi–arithmetic mean preserves α–irreflexivity of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Definition 7 (Calvo et al. 2002)**.** A function $F : [0,1]^n \to [0,1]$ is additive, if

$$\forall_{i=1,...,n} \ \forall_{x_i, y_i, x_i+y_i \in [0,1]} \ F(x_1+y_1, ..., x_n+y_n) = F(x_1,...,x_n) + F(y_1,...,y_n).$$

Example 4. Weighted arithmetic means are additive functions.

Theorem 7. *Let $\alpha_1, ..., \alpha_n \in [0,1]$, $F : [0,1]^n \to [0,1]$ be a super additive aggregation function. If relations $R_i \in FR(X)$ are α_i–irreflexive for $i = 1,...,n$, then relation $R = F(R_1,...,R_n)$ is α–irreflexive, for $\alpha = F(\alpha_1,...,\alpha_n)$.*

Theorem 8. *Let* $\alpha \in [0,1]$, $F \geqslant \max$. *If* $R = F(R_1, \ldots, R_n)$ *is* α*-irreflexive, then all relations* $R_1, ..., R_n$ *are* α*-irreflexive.*

Example 5. Let $card\, X = 2$. We consider fuzzy relations with matrices:

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$T_1 = \min(R,S) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, T_2 = \frac{R+S}{2} = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

Relation T_1 is α-irreflexive for $\alpha \in [0,1]$, T_2 for $\alpha \in [0, 0.5]$, but relations R, S do not have this property for any $\alpha \in [0,1]$.

Asymmetry

Definition 8 (Drewniak 1989)**.** Let $\alpha \in [0,1]$. $R \in FR(X)$ is:

- α–asymmetric, if

$$\underset{x,y \in X}{\forall} \min(R(x,y), R(y,x)) \leqslant 1-\alpha,$$

- α–antisymmetric, if

$$\underset{x,y,x \neq y \in X}{\forall} \min(R(x,y), R(y,x)) \leqslant 1-\alpha.$$

Theorem 9. *Let $\alpha \in [0,1]$, card $X \geqslant 2$. $F : [0,1]^n \to [0,1]$ preserves α–asymmetry (α–antisymmetry) of fuzzy relations, iff*

$$\underset{s,t \in [0,1]^n}{\forall} (\underset{1 \leqslant k \leqslant n}{\forall} \min(s_k, t_k) \leqslant 1-\alpha) \Rightarrow \min(F(s), F(t)) \leqslant 1-\alpha.$$

Theorem 10. *Let card $X \geqslant 2$. $F : [0,1]^n \to [0,1]$ preserves α–asymmetry (α–antisymmetry) of fuzzy relations for arbitrary $\alpha \in [0,1]$, iff*

$$\underset{s,t \in [0,1]^n}{\forall} \min(F(s), F(t)) \leqslant \max_{1 \leqslant k \leqslant n} \min(s_k, t_k).$$

Corollary 3. *The median function and minimum preserve α–asymmetry (α–antisymmetry) of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Example 6. Let *card* $X = 2$. $R, S, T \in FR(X)$, $F(s,t) = \max(s,t)$,

$$R = \begin{bmatrix} 0 & 0.8 \\ 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 0.8 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $T = F(R, S)$. R, S are α–asymmetric (α–antisymmetric) for $\alpha \in [0, 0.2]$ and T does not have this property for any $\alpha \in [0, 1]$.

Theorem 11. *Let $\alpha \in [0, 1]$, $F \geqslant \max$. If $R = F(R_1, \ldots, R_n)$ is α-asymmetric (antisymmetric), then all relations $R_1, ..., R_n$ are α-asymmetric (antisymmetric).*

Example 7. Let $\alpha_1, ..., \alpha_n \in [0, 1]$. If relations $R_i \in FR(X)$ are α_i–asymmetric (α_i–antisymmetric) for $i = 1, ..., n$, then relation $R \in FR(X)$ is α–asymmetric (α–antisymmetric), where

$$R = \frac{1}{n} \sum_{i=1}^{n} R_i, \qquad \alpha = \frac{1}{n} \min_{1 \leqslant i \leqslant n} \alpha_i.$$

Example 8. Let $\alpha_1, ..., \alpha_n \in [0, 1]$. If relations $R_i \in FR(X)$ are α_i–asymmetric (α_i–antisymmetric) for $i = 1, ..., n$, then relation $R \in FR(X)$ is α–asymmetric (α–antisymmetric), where

$$R = \min(R_1, ..., R_n), \qquad \alpha = \min_{1 \leqslant i \leqslant n} \alpha_i.$$

Connectedness

Definition 9 (Drewniak 1989)**.** Let $\alpha \in [0,1]$. $R \in FR(X)$ is:

- totally α–connected, if

$$\forall_{x,y \in X} \ \max(R(x,y), R(y,x)) \geqslant \alpha,$$

- α–connected, if

$$\forall_{x,y,x \neq y \in X} \ \max(R(x,y), R(y,x)) \geqslant \alpha.$$

Theorem 12. *Let $\alpha \in [0,1]$, card $X \geqslant 2$. $F : [0,1]^n \to [0,1]$ preserves total α–connectedness (α–connectedness) of fuzzy relations, iff*

$$\forall_{s,t \in [0,1]^n} \ (\forall_{1 \leqslant k \leqslant n} \ \max(s_k, t_k) \geqslant \alpha) \Rightarrow \max(F(s), F(t)) \geqslant \alpha.$$

Theorem 13. *Let card $X \geqslant 2$. $F : [0,1]^n \to [0,1]$ preserves total α–connectedness (α–connectedness) of fuzzy relations for arbitrary $\alpha \in [0,1]$, iff*

$$\forall_{s,t \in [0,1]^n} \ \max(F(s), F(t)) \geqslant \min_{1 \leqslant k \leqslant n} \max(s_k, t_k).$$

Corollary 4. *Maximum and the median preserve total α–connectedness (α–connectedness) of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Example 9. Let *card* $X = 2$. $R, S, T \in FR(X)$, $F(s,t) = \min(s,t)$,

$$R = \begin{bmatrix} 1 & 0.8 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0.8 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $T = F(R,S)$. R, S are totally α–connected (α–connected) for $\alpha \in [0, 0.8]$ and T does not have this property for any $\alpha \in [0,1]$.

Theorem 14. *Let $\alpha \in [0,1]$, $F \leqslant \min$. If $R = F(R_1, \dots, R_n)$ is totally α-connected (α-connected), then all relations $R_1, ..., R_n$ are totally α-connected (α-connected).*

Example 10. Let $\alpha_1, ..., \alpha_n \in [0,1]$. If relations $R_i \in FR(X)$ are α_i–connected (totally α_i–connected) for $i = 1, ..., n$, then relation $R \in FR(X)$ is α–connected (totally α–connected), where

$$R = \frac{1}{n}\sum_{i=1}^{n} R_i, \qquad \alpha = \frac{1}{n}\max_{1 \leqslant i \leqslant n} \alpha_i.$$

Example 11. Let $\alpha_1, ..., \alpha_n \in [0,1]$. If relations $R_i \in FR(X)$ are α_i–connected (totally α_i–connected) for $i = 1, ..., n$, then relation $R \in FR(X)$ is α–connected (totally α–connected), where

$$R = \max(R_1, ..., R_n), \qquad \alpha = \max_{1 \leqslant i \leqslant n} \alpha_i.$$

Symmetry

Definition 10 (Drewniak 1989)**.** Let $\alpha \in [0,1]$. Relation $R \in FR(X)$ is α–symmetric, if

$$\forall_{x,y \in X} \; R(x,y) \geqslant 1-\alpha \Rightarrow R(y,x) \geqslant R(x,y).$$

Theorem 15. *Let $\alpha \in [0,1]$. If $F : [0,1]^n \to [0,1]$ fulfils*

$$F|_{[0,1]^n \setminus [1-\alpha,1]^n} < 1-\alpha,$$

then it preserves α–symmetry of relations $R_1, ..., R_n \in FR(X)$.

Theorem 16. *If a function $F : [0,1]^n \to [0,1]$ fulfils condition $F \leqslant \min$ then it preserves α–symmetry of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Corollary 5. *Any t-norm preserves α–symmetry of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Example 12. Since any projection P_k, $k \in \mathbb{N}$, preserves the α–symmetry for each $\alpha \in [0,1]$ but it is not true that $P_k \leqslant \min$, then Theorem 16 gives only a sufficient condition for preservation of the α–symmetry for any $\alpha \in [0,1]$.

Example 13. Let $card\, X = 2$. We consider fuzzy relations with matrices:

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$T_1 = \min(R, S) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T_2 = \max(R, S) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$T_3 = \frac{R+S}{2} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}.$$

Relations T_1, T_2, T_3 are α-symmetric for $\alpha \in [0, 1]$, but relations R, S do not have this property for any $\alpha \in [0, 1]$.

Transitivity

Definition 11. Let $\alpha \in [0,1]$. Relation $R \in FR(X)$ is α–transitive, if

$$\forall_{x,y,z \in X} \min(R(x,y), R(y,z)) \geqslant 1-\alpha \Rightarrow R(x,z) \geqslant \min(R(x,y), R(y,z)).$$

Definition 12 (Saminger et al. 2002)**.** Let m, $n \in \mathbb{N}$. Operation $F : [0,1]^m \to [0,1]$ dominates operation $G : [0,1]^n \to [0,1]$ ($F \gg G$), if for arbitrary matrix $[a_{ik}] = A \in [0,1]^{m \times n}$ we have

$$F(G(a_{11}, ..., a_{1n}), ..., G(a_{m1}, ..., a_{mn})) \geqslant G(F(a_{11}, ..., a_{m1}), ..., F(a_{1n}, ..., a_{mn})).$$

Theorem 17. *Let $\alpha \in [0,1]$. If increasing $F : [0,1]^n \to [0,1]$ fulfils*

$$F|_{[0,1]^n \setminus [1-\alpha,1]^n} < 1-\alpha,$$

and $F \gg \min$, then it preserves α–transitivity of fuzzy relations.

Example 14. Let $a \in (0,1]$ and $F : [0,1]^2 \to [0,1]$ be of the form

$$F(s,t) = \begin{cases} 0, & (s,t) \in [0,a) \times [0,a) \\ \min(s,t), & \text{otherwise} \end{cases}$$

F is a t–norm and $F|_{[0,1]^n \setminus [1-\alpha,1]^n} < 1 - \alpha$ but it does not dominate minimum. However, F preserves the α–transitivity for each $\alpha \in [0,1)$ and $\alpha \leqslant 1 - a$. As a result conditions for preservation of the α–transitivity stated in Theorem 17 are only sufficient.

Theorem 18. *If a function $F : [0,1]^n \to [0,1]$ is increasing in each variable, fulfils $F \gg \min$ and $F \leqslant \min$ then it preserves α–transitivity of fuzzy relations for any $\alpha \in [0,1]$.*

Corollary 6. *Minimum and the aggregation function*

$$A_w(t_1, \ldots, t_n) = \begin{cases} 1, & (t_1, \ldots, t_n) = (1, \ldots, 1) \\ 0, & \textit{otherwise} \end{cases}$$

preserve the α–transitivity of fuzzy relations for any $\alpha \in [0,1]$ because both functions fulfil assumptions of Theorem 18.

References

[1] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer, Berlin 2007.

[2] T.Calvo, A. Kolesarova, M. Komornikowa, R. Mesiar, Aggregation operators: Properties, classes and construction methods, in: Aggregation Operators, T. Calvo et al. (Eds.), Physica-Verlag, Heildelberg, 2002, 3–104.

[3] J. Drewniak, Fuzzy relation calculus, Silesian University, Katowice, 1989.

[4] J. Drewniak, U. Dudziak, Preservation of properties of fuzzy relations during aggregation processes, Kybernetika **43** (2) (2007), 115-132.

[5] U. Dudziak, Graded properties of fuzzy relations in aggregation process, Journal of Electrical Engineering **56** (2005) (12/s), 56–58.

[6] J.L. Garcia-Lapresta, L.C. Meneses, Individual-valued preferences and their aggregation: consistency analysis in a real case, Fuzzy Sets Syst. **151** (2005), 269–284.

[7] M. Grabisch, E. Pap, R. Mesiar, J.-L. Marichal, Aggregation Functions, Cambridge University Press, Cambridge 2009.

[8] G.H. Hardy, J. Littlewood, G. Polya, Inequalities, Cambridge University Press, Cambridge, 1955.

[9] J. Fodor, M. Roubens, Fuzzy preference modelling and multicriteria decision support, Kluwer Acad. Publ., Dordrecht, 1994.

[10] E.P. Klement, R. Mesiar, E.Pap, Triangular norms, Kluwer Acad. Publ., Dordrecht, 2000.

[11] V. Peneva, I.Popchev, Aggregation of fuzzy relations, C. R. Acad. Bulgare Sci. **51**(9-10) (1998), 41–44.

[12] V. Peneva, I. Popchev, Properties of the aggregation operators related with fuzzy relations, Fuzzy Sets Syst. **139** (3) (2003), 615–633.

[13] S. Saminger, R. Mesiar, U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, Int. J. Uncertain., Fuzziness, Knowl.-Based Syst. **10**, Suppl. (2002), 11–35.

[14] L.A. Zadeh, Fuzzy sets, Inform. Control **8** (1965), 338–353.

[15] L.A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. **3** (1971), 177-200.