The Stucture of Qualitative Fuzzy Measures on Finite Scales

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Motivation

- Numerical fuzzy measures (capacities) are monotonic set-functions that subsume many kinds of set functions used in uncertainty modelling, game theory and multicriteria analysis.
- For instance: coherent imprecise probabilities, 2-monotone functions, n-monotone functions, belief functions, probability, possibility and necessity measures.
- Qualitative fuzzy measures ranging on a finite totally ordered scale are less well-known.
- Replacing addition by maximum, possibility measures seem to be the counterpart of probability measures.
- This talk discusses to what extent the classification in terms of belief function and upper/lower probabilities carries over to qualitative fuzzy measures and possibility theory.

A missing notion?

	Quantitative	**Qualitative**
• *Preference aggregation*		
	– weighted sum	weighted min and max
	– Choquet integral	Sugeno integral
• *Uncertainty modeling*		
	– probability theory	possibility theory
	– Shafer evidence theory	?
	– imprecise probability	

OUTLINE

1. The numerical setting : known results
2. The qualitative setting : q-capacities generated by basic possibility assignments
3. Information comparison for q-capacities
4. q-capacities as families of possibility measures
5. Relations between q-capacities and modal logic

Monotonic set functions

A *capacity* (or fuzzy measure) on a finite space $S = \{s_1, \ldots, s_{|S|}\}$ is a mapping $\gamma : 2^S \to L$, and L is a chain with top 1 and bottom 0, such that

- $\gamma(\emptyset) = 0; \gamma(S) = 1;$
- If $A \subseteq B$ then $\gamma(A) \leq \gamma(B)$

Numerical capacities :

- $L = [0, 1]$.
- The conjugate γ^c of γ is a -capacity $\gamma^c(A) = 1 - \gamma(A^c), \forall A \subseteq S$, where A^c is the complement of set A.

Qualitative capacities :

- $L = \{\lambda_0 = 0 < \lambda_1 \cdots < \lambda_n = 1\}$, a finite chain equipped with $\min, \max$, and an involutive order-reversing map ν.
- The conjugate γ^c of q-capacity γ is a q-capacity defined by $\gamma^c(A) = \nu(\gamma(A^c)), \forall A \subseteq S$.

Possibility and necessity measures

- A special case of q-capacity is a possibility measure : $\Pi(A) = \max_{s \in A} \pi(s)$.
  - The possibility distribution $\pi : \pi(s) = \Pi(\{s\})$ is enough to recover the set-function
  - The value $\pi(s)$ is understood as the possibility that s be the actual state of the world: $\exists s \in S : \pi(s) = 1$.
- The characteristic property of possibility measures is *maxitivity*: $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$
- Another special case of q-capacity is the necessity measure such that $N(A \cap B) = \min(N(A), N(B))$.
  - They are such that $N(A) = \min_{s \notin A} N(S \setminus \{s\})$ where $\iota(s) = N(S \setminus \{s\})$ is a degree of impossibility of s.
  - The conjugate of a possibility measure Π is a necessity measure $N(A) = \nu(\Pi(A^c))$ such that $\iota(s) = \nu(\pi(s))$.

Belief functions formalism

Basic probability assignment (bpa)

- bpa: probability function on 2^S. $m : 2^{|S|} \to [0,1]$ s.t. $m(\emptyset) = 0$ and $\sum_{E \subseteq S} m(E) = 1$
- a set E with positive mass $m(E) > 0$ is a focal set

Two set-functions: Belief, Plausibility

- Belief: $bel(A) = \sum_{E \subseteq A} m(E)$ (a capacity)
- Plausibility: $pl(A) = \sum_{E: A \cap E \neq \emptyset} m(E) = 1 - bel(A^c)$ is the conjugate of bel.

These set-functions are in 1-to-1 correspondence with each other, and with the bpa m.

The bpa is called the Moebius transform of Bel

Contour functions, probability and possibility measures

- Given bpa m, its contour function $\pi_m : S \rightarrow [0,1]$ is

$$\pi_m(s) = pl(\{s\}) = \sum_{s \in A} m(A)$$

- *If all focal sets are singletons, $bel = pl =$ probability measure with probability distribution π_m*
- *Consonance*: The support of m is a family of nested sets if and only if $pl(A) = \max_{s \in A} pl(\{s\})$ (pl *is a possibility measure with possibility distribution* π_m). Then the conjugate belief function is a necessity measure.
- *Refining possibility by probability*: Given a (qualitative or quantitative) possibility measure with distribution π, there exists a super-increasing mapping $\phi : L \rightarrow [0,1]$ where $\pi(s) \mapsto p(s) = \phi(\pi(s))$, where p is a big-stepped probability distribution ($\forall s \in S, p(s) > \sum_{u \in S: p(s) > p(u)} p(u)$) and

$$\Pi(A) > \Pi(B) \Rightarrow P(A) > P(B)$$

Upper and lower probabilities

- *Belief function as a probability family*: a bpa m induces a convex non-empty core $\mathcal{P}_m = \{P | \forall A \subset S,\ Bel(A) \leq P(A) \leq Pl(A)\}$
- More generally the core $\mathcal{P}_g = \{P | \forall A \subset S,\ P(A) \geq g(A)\}$ for a capacity g may be empty. If not, it is a convex probability set.
- A sufficient condition for non empty-core is super-modularity : g is a convex capacity: $g(A \cup B) + g(A \cap B) \geq g(A) + g(B)$.
- Order n-super-modularity does not imply $n + 1$-supermodularity.
- belief functions are exactly order ∞-super-modular capacities.
- Not all convex sets of probabilities can be described by capacities (need lower expectations).
- Coherent capacities g characterize some convex sets $\mathcal{P}_g$: they are such that $g(A) = \inf\{P(A), P \in \mathcal{P}_g\}$, for instance, convex capacities, belief functions, necessity measures.

Qualitative capacities from basic possibilistic assignments

Basic possibility assignment (Dubois Prade, 1983)

- bπa: possibility distribution $\mu : 2^S \to L$ s.t. $\mu(\emptyset) = 0$ and $\max_{E \subseteq S} \mu(E) = 1$
- a set E with positive mass $\mu(E) > 0$ is a focal set

Two set-functions generalizing possibility measures : $s \in A$ can become $E \subseteq A$ or $A \cap E \neq \emptyset$ like for belief functions.

- Lower Possibility measure: $\Pi_*(A) = \max_{E \subseteq A} \mu(E)$
- Upper Possibility measure: $\Pi^*(A) = \max_{E: A \cap E \neq \emptyset} \mu(E) \geq \Pi_*(A)$.

Remarks

1. If focal sets are singletons, then $\Pi_* = \Pi^* = \Pi$ is a possibility measure
2. If focal sets are nested then Π_* is a necessity measure
3. Upper and lower possibility measures are NOT conjugate to each other : $\max(\Pi_*(A), \Pi^*(A^c)) = 1$, but $\Pi_*(A) \neq \nu(\Pi^*(A^c))$.

Capacities as lower possibility functions

- Given bπa μ, the contour function $\pi_\mu : S \to L$ is

$$\pi_\mu(s) = \Pi^*(\{s\}) = \max_{A: s \in A} \mu(A)$$

- **Properties**
  - The upper possibility measure is always a possibility measure with distribution π_μ : $\Pi^*(A) = \max_{s \in A} \pi_\mu(s)$
  - **A lower qualitative possibility measure is a general q-capacity, and any q-capacity is a lower possibility measure**:
    if μ is increasing w.r.t inclusion, then $\Pi_*(A) = \mu(A)$.
  - Contrary to the numerical setting, there is **not** a 1-to-1 correspondence between general capacities and bπa's.

Basic possibility assignments generating a q-capacity

- Define an equivalence relation $\equiv$ on the set $\mathcal{M}$ of bπa's as follows:

$$\mu_1 \equiv \mu_2 \iff \Pi_*^1 = \Pi_*^2$$

  where $\Pi_*^i(A) = \max_{E \subseteq A} \mu_i(E), \forall A \subseteq S$

- Let $C_\gamma = \{\mu_i | \Pi_*^i = \gamma\} \in \mathcal{M}/\equiv$
- **Proposition** :
  1. C_γ has a greatest element $\mu = \gamma$
  2. C_γ has a least element $\gamma_\#$ known as as qualitative Moebius transform :

$$\begin{aligned} \gamma_\#(E) &= \gamma(E) \quad \text{if } \gamma(E) > \max_{B \subsetneq E} \gamma(B) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

  3. $C_\gamma = \{\mu | \gamma_\# \leq \mu \leq \gamma\}$

- The choice of $\mu \in C_\gamma$ affects the contour function hence the upper possibility measure: $\Pi_i^* \geq \gamma = \Pi_*^i, \forall \mu_i \in C_\gamma$.

Inner (qualitative) Moebius transforms

- We call $\gamma_{\#}$ inner (qualitative) Moebius transform because there is an outer one based on supersets. Due to Mesiar and Grabisch (1997)
- Can be written as $\gamma_{\#}(E) = \gamma(E) \ominus \max_{s \in E} \gamma(E \setminus s)$, where $a \ominus b = \min\{c | \max(b, c) \geq a\}$
- They are bπa such that if $A \subset B$ and $\gamma_{\#}(A) > 0$ then $\gamma_{\#}(B) > \gamma_{\#}(A)$ strictly monotonic with inclusion on $\mathcal{F}^{\gamma}$.
- The inner (qualitative) Moebius transform of the q-capacity γ contains the minimal information needed to reconstruct it since, by construction

$$\gamma(A) = \max_{E \subseteq A} \gamma_{\#}(E)$$

  so there is a bijection between capacities γ and bπ's of the form $\gamma_{\#}$.
- $\mathcal{F}^{\gamma} = \{E, \gamma_{\#}(E) > 0\}$ is the family of focal sets associated to γ.

Inner Moebius transforms of possibility measures

Inner (qualitative) Moebius transforms $\gamma_\#$ are special bπa's that are inclusion-monotonic on their support, and generalised possibility distributions (on 2^S)

- $\gamma_\#(\emptyset) = 0$; $\gamma_\#(A) = 1$ for some $A \neq \emptyset$.
- From $\max_{s \in A} \pi(s)$ to $\max_{E \subseteq A} \gamma_\#(E)$.
- The inner qualitative Moebius transform of a possibility measure coincides with its possibility distribution: $\Pi_\#(A) = \pi(s)$ if $A = \{s\}$ and 0 otherwise
- $\mathcal{F}^\Pi = \{\{s\}, \pi(s) > 0\}$ (similar to probability measures in numerical setting)

Inner Moebius transforms of necessity measures

The inner qualitative Moebius transform of a necessity measure has nested focal sets.

- $N(A \cap B) = \min(N(A), N(B)) \iff \forall E, F \in \mathcal{F}^N, E \subset F$ or $E \subset F$
- The cut-set $\{B | N(B) \geq \lambda\}$ is a proper filter (deductively closed under inclusion and intersection) : it has a single least element $E_\lambda \neq \emptyset$.
- $E \in \mathcal{F}^N$ if and only if $\lambda > 0 \in L, N(E) = \lambda$ and $E = \cap\{B | N(B) \geq \lambda\}$
- $\mathcal{F}^N = \{E_\lambda, \lambda > 0\}$ and $N_\#(E) = \min\{\lambda | E = E_\lambda\}$
- $\mathcal{F}^N$ contains the $\nu(\lambda)$-cuts of the possibility distribution of the conjugate $\Pi = N^c$ with possibility distribution

$$\pi(s) = \min_{s \notin E} \nu(N_\#(E)) = \nu(\lambda_s)$$

where $\lambda_s = \max\{N_\#(E) | s \notin E\} = N(S \setminus \{s\})$

Three points of view on qualitative capacities

- **Like Dempster**: upper and lower possibility functions induced a possibility distribution π on a set W and a multimapping $\Gamma : W \to 2^S$ (Dubois Prade 1985; De Baets Tsiporkova 1997) :
  - $\Pi_*(A) = \Pi(\{w : \Gamma(w) \subseteq A\})$ is a q-capacity;
    $\Pi^*(A) = \Pi(\{w : \Gamma(w) \cap A \neq \emptyset\})$ is a possibility measure
  - The interval $[\Pi_*(A), \Pi^*(A)]$ contains the real value of $\Pi(A)$ induced by the real selection $f \in \Gamma$.
  - The basic information is $(W, \pi), \Gamma : W \to 2^S$
- **Like Shafer** : $qbel(A) = \max_{E \subseteq A} \mu(E)$; $qpl(A) = \max_{E \cap A \neq \emptyset} \mu(E)$
  - the basic information is a bπa μ from which the pair (Bel, Pl) is induced.
  - The pair $(qbel, qpl)$ is not enough to recover μ
- **Like Walley** : The basic information is the q-capacity. What is the bridge with families of possibility measures ?

Informational comparison of capacities

One may characterise the relative position of two capacities γ_1 and γ_2 in terms of bπa's

- $\gamma_1 \sqsubseteq \gamma_2 : \forall A' \in \mathcal{F}^{\gamma_2}\ \exists A \in \mathcal{F}^{\gamma_1}$ s. t. $A \subseteq A'$ and $\gamma_{2\#}(A') \leq \gamma_{1\#}(A)$.
- **Proposition** : $\gamma_1 \sqsubseteq \gamma_2$ if and only if $\gamma_2 \leq \gamma_1$.
- A similar result holds for belief functions : A bpa m_1 is a *specialization* of a bpa m_2 ($m_1 \sqsubseteq m_2$) if and only if
  - Any focal set of m_2 contains at least one focal set of m_1.
  - Any focal set of m_1 is included in at least one focal set of m_2
  - $m_2(F_j) = \sum_i w_{ij} \cdot m_1(E_i)$, with constraint $w_{ij} > 0$ only if $E_i \subseteq F_j$.
- $m_1 \sqsubseteq m_2$ implies $bel_2 \leq bel_1$ and $pl_1 \leq pl_2$ (not the converse).
- m_1 is then more informative than m_2

Is it still the case for q-capacities constructed from bπa's (q-belief functions)??

Informational comparison of capacities

- What does $\gamma_2 \leq \gamma_1$ mean? is γ_2 more or less informative than γ_1? Not clear since $\gamma_2 \leq \gamma_1 \iff \gamma_1^c \leq \gamma_2^c$, where $\gamma^c(A) = \nu(\gamma(A^c))$.
- A possibility measure Π_1 is said to be more informative (specific) than another one Π_2 if $\forall A \subset S, \Pi_1(A) \leq \Pi_2(A)$ (equivalently $\forall s \in S, \pi_1(s) \leq \pi_2(s)$).
  - In the case of possibility measures, $\Pi_2 \leq \Pi_1$ means that Π_2 is more informative than Π_1 ($\Pi_?(A) = 1, \forall A \neq \emptyset$: total ignorance)
  - In the case of necessity measures, $N_2 \leq N_1$ means that N_2 is less informative than N_1 ($N_?(A) = 0, \forall A \neq S$: total ignorance)
- The only way to make sense of the eventwise comparison is to qualify a q-capacity measure γ in terms of its optimism or pessimism
  - For instance the vacuous $\Pi_?$ is an optimistic representation of ignorance, while the vacuous $N_?$ is a pessimistic view of the same information state.
  - in the numerical case, bel is always pessimistic when induced by a mass function; this is not so for q-capacities since bπa's generate all of them.

Optimistic and pessimistic q-capacities

The above discussion leads us to

- define optimistic and pessimistic q-capacities based on conjugate pairs
- restrict information comparison $\sqsubseteq$ to sets of focal sets inducing pessimistic q-capacities
- show that $\gamma_2 \leq \gamma_1$ means that γ_2 is less informative than γ_1 if these q-capacities are pessimistic.

A q-capacity γ is said to be pessimistic (resp. optimistic) if $\gamma \leq \gamma^c$ (resp. if $\gamma \geq \gamma^c$).

- a q-capacity can be neither. There may exist A, B such that $\gamma(A) < \gamma^c(A)$, and $\gamma(B) > \gamma^c(B)$.
- a q-capacity can be both : $\gamma = \gamma^c$ is possible. For instance, on a space with $2n + 1$ elements, $\gamma_n(A) = 1$ if $|A| > n$ and 0 otherwise.

Q-capacities having the same information content

Given a capacity γ, one can derive its pessimistic and optimistic counterparts:

$$\gamma_*(A) = \min(\gamma(A), \gamma^c(A))$$

$$\gamma^*(A) = \max(\gamma(A), \gamma^c(A)).$$

By construction, γ_* is pessimistic and γ^* is optimistic.

- γ_* and γ^* are capacities.
- γ is more optimistic than γ_* and less optimistic than γ^*.
- They have the same information content :

Indeed, the actual information about a set A is given by $\{\gamma(A), \gamma(A^c)\}$ that in general is not redundant. And it is clear that $\{\gamma_*(A), \gamma_*(A^c)\}$, $\{\gamma^*(A), \gamma^*(A^c)\}$ and $\{\gamma(A), \gamma(A^c)\}$ contain the same information.

Degrees of pessimism and optimism of a q-capacity

γ and δ contain the same amount of information (denoted by $\gamma \approx \delta$) if and only if $\gamma^* = \delta^*$ and $\gamma_* = \delta_*$.

- An equivalence class contains q-capacities that only differ by their amount of optimism.
- The equivalence class of γ is clearly upper bounded by γ^* and lower-bounded by γ_*.
- The degree of optimism of γ can be evaluated as

$$opt(\gamma) = \frac{|\{A \subset \Omega : A \neq \emptyset, \gamma(A) \geq \gamma^c(A)\}|}{2^{|\Omega|} - 2}.$$

- The degree of optimism of a possibility measure is 1, The degree of optimism of a necessity measure is 0. More generally, $opt(\gamma^*) = 1$.

Back to information content comparison

A q-capacity γ is said to be to be more informative than a q-capacity δ if and only if $\gamma_* \geq \delta_*$.

- $\gamma_*(A)$ can be interpreted in terms of degree of certainty rather than plausibility: $\forall A \subseteq \Omega \min(\gamma_*(A), \gamma_*(A^c)) < 1$;
- in fact, if L has n elements then, $\min(\gamma_*(A), \gamma_*(A^c)) \leq \lambda_p$ or $\leq \lambda_{p+1}$, according to whether $n = 2p$ or $2p + 1$. Moreover if $\gamma_*(A) = 1$ then $\gamma_*(A^c) = 0$. And we may have $\gamma_*(A) = \gamma_*(A^c) = 0$.
- A necessity measure expresses certainty and is pessimistic.
- The formal analogy of q-capacities (of the form $\gamma(A) = \max_{B \subseteq A} \gamma_{\#}(B)$) with belief functions makes better sense if the q-capacity is pessimistic.

Capacities as directed posets of possibility functions

- Possibilistic core of γ: $\mathcal{R}(\gamma) = \{\pi : \Pi(A) \geq \gamma(A), \forall A \subseteq S\}$.
- $\mathcal{R}(\gamma) \neq \emptyset$: There is always at least one possibility measure that dominates any q-capacity: the vacuous possibility measure $\Pi(A) = 1, \forall A \neq \emptyset$.
- Clearly, if Π_1 and Π_2 are possibility measures, then $\max(\Pi_1, \Pi_2)$ is a possibility measure too (less specific than both Π_1 and Π_2)
- Π_1 and Π_2 are in $\mathcal{R}(\gamma)$ then so is $\max(\Pi_1, \Pi_2)$.
- So, $\mathcal{R}(\gamma)$ is a directed poset of possibility functions

So, we should try to find the minimal elements in set $\mathcal{R}(\gamma)$ (the most specific possibility measures)

Capacities as Lower Possibilities : permutations

So, *any qualitative q-capacity is a lower possibility measure*:

$$\gamma(A) = \min_{\pi \in \mathcal{R}(\gamma)} \Pi(A).$$

But we can restrict to least elements in $\mathcal{R}(\gamma)$.

- Let σ be a permutation of the $n = |S|$ elements in S. $S_\sigma^i = \{s_{\sigma(i)}, \ldots, s_{\sigma(n)}\}$. Define the possibility distribution π_σ^γ :

$$\forall i = 1 \ldots, n, \pi_\sigma^\gamma(s_{\sigma(i)}) = \gamma(S_\sigma^i)$$

- **Results**: We can find the least elements among the π_σ^γ's
  - $\forall A \subseteq S, \Pi_\sigma^\gamma(A) \geq \gamma(A)$.
  - $\forall A \subseteq S, \gamma(A) = \min_\sigma \Pi_\sigma^\gamma(A)$
  - $\forall \pi \in \mathcal{R}(\gamma), \pi(s) \geq \pi_\sigma^\gamma(s), \forall s \in S$ for some permutation σ of S.

There are at most $n!$ elements thus obtained: still too many.

Capacities as Lower Possibilities : selections

- A selection function $sel : \mathcal{F}^\gamma \to S$ assigns to each focal subset $A \in \mathcal{F}^\gamma$ one element $s = sel(A) \in A$.
- We can assign to each selection function a possibility distribution π^γ_{sel} by letting $\max \emptyset = 0$ and

$$\pi^\gamma_{sel}(s) = \max_{E: sel(E) = s} \gamma_\#(E), \forall s \in S.$$

- If $\gamma = \Pi$, then there is only one possible selection function and $\pi^\Pi_{sel} = \pi$.
- **Results**: We can find the least elements among the π^γ_{sel}'s
  - For any selection function sel with domain $\mathcal{F}^\gamma$ it holds that $\forall A \subseteq S, \Pi^\gamma_{sel}(A) \geq \gamma(A)$.
  - $\forall A \subseteq S$, $\gamma(A) = \min_{sel \in \Sigma(\mathcal{F}^\gamma)} \Pi^\sigma_{sel}(A)$.
  - Now we have $\prod_{E \in \mathcal{F}^\gamma} |E|$ possibility distributions. Still too many.

Useful selection functions

The set of minimal elements (maximally specific) of $\mathcal{R}(\gamma)$ is also included in $\{\pi^\gamma_{sel}, sel \in \Sigma(\mathcal{F}^\gamma)\}$.

More generally the useful selection functions can be defined as follows:

Algorithm MSUP Maximal specific upper possibility generation

1. Rank the focal sets E_j in decreasing order of $\gamma_\#(E_j)$. Let $j = 1$ and $\mathcal{F} = \mathcal{F}^\gamma$.
2. Define $sel(E_j) = s_j$ for some $s_j \in E_j$ and let $\pi(s_j) = \gamma_\#(E_j)$. Delete E_j from $\mathcal{F}$.
3. $\forall k$ such that $s_j \in E_k$ for some other $E_k \in \mathcal{F}$ (it is such that $\gamma_\#(E_j) \geq \gamma_\#(E_k)$) let $sel(E_k) = s_j$, then delete E_k from $\mathcal{F}$.
4. Repeat from step 2 until $\mathcal{F} = \emptyset$

$\Sigma_*(\gamma) =$ selection functions generated by algorithm MSUP, called *useful selection functions*, and $\mathcal{R}_*(\gamma)$ be the corresponding set of possibility distributions.

Minimally specific dominating possibilities

A selection function $sel \in \Sigma_*(\gamma)$ satisfies the following property : If $sel(E) = s$ for some $E \in \mathcal{F}^\gamma$ and $\pi_{sel}(s) = \gamma_\#(E)$, then $\forall F \in \mathcal{F}^\gamma$, such that $s \in F$:

- if $\gamma_\#(E) \geq \gamma_\#(F)$ then $sel(F) = s$
- if $\gamma_\#(E) < \gamma_\#(F)$ then $sel(F) \neq s$.

Results:

- If $\pi \neq \rho \in \mathcal{R}_*(\gamma)$, then neither $\pi > \rho$ nor $\pi > \rho$ hold.
- For any permutation σ, there exists a selection function sel corresponding to another permutation τ such that $\pi_\sigma \geq \pi_\tau = \pi_{sel}$.
- $\mathcal{R}_*(\gamma)$ is the set of maximally specific possibility distributions such that $\Pi(A) \geq \gamma(A)$ (the extreme points of the qualitative core).

Note on qualitative contour functions

- $\pi_\gamma(s) = \max_{s\in E} \gamma_\#(E)$ is the qualitative contour function of γ (compare with $\pi_m(s) = \sum_{s\in E} m(E)$ for belief functions)
- Moreover, $\forall A, \Pi_\gamma(A) = \max_{A\cap E\neq\emptyset} \gamma_\#(E)$ (this is the upper possibility function induced by $\gamma_\#$).
- Note: here γ is the primitive information.

Proposition: $\pi_\gamma(s) = \max_{\pi\in\mathcal{R}_*(\gamma)} \pi(s)$.
We also have $\Pi_\gamma(A) = \max_{\pi\in\mathcal{R}_*(\gamma)} \Pi(A)$, while $\gamma(A) = \min_{\pi\in\mathcal{R}_*(\gamma)} \Pi(A)$.

Remark: We cannot use the necessity measure induced by the contour function of γ as a lower bound for the latter. Indeed

$$N_\gamma(A) = \nu(\Pi_\gamma(A^c)) = \min_{s\notin A} \nu(\max_{s\in E} \gamma_\#(E)) = \min_{E\cap A^c\neq\emptyset} \nu(\gamma_\#(E))$$

which cannot be compared with $\gamma(A)$.

Capacities as upper necessities

A totally dual construction can be developed using outer qualitative Moebius transforms:

$$\begin{aligned} \gamma^{\#}(A) &= \gamma(A) \text{ if } \gamma(A) < \min\{\gamma(F), A \subset F\} \\ &= 1 \text{ otherwise.} \end{aligned}$$

Denote by γ^c the conjugate of γ: $\gamma^c(A) = \nu(\gamma(A^c)), \forall A \subseteq S$, where A^c is the complement of set A, and ν the order-reversing map on L.

The inner qualitative Moebius transform $\gamma^c_{\#}$ of γ^c is related to the outer qualitative mass function $\gamma^{\#}$ (Dubois Fargier, 2009):

$$\gamma^{\#}(E) = \nu(\gamma^c_{\#}(E^c)).$$

Results :

- $\gamma(A) = \max\{N(A), \pi \in \mathcal{R}_*(\gamma^c)\}$ (any q-capacity is an upper necessity measure)
- $\gamma(A) \geq N_{\gamma_c}(A) = \min_{s \notin A} \iota_\gamma(s)$. where $\iota_\gamma(s) = \nu(\pi_{\gamma_c}(s))$ (anti-contour function)

Sugeno Integral as lower possibilistic integral

Let $f : S \to L$ be a function that may serve as a utility function if S is a set of attributes. Sugeno integral is often defined as follows:

$$\mathcal{S}_\gamma(f) = \max_{\lambda \in L} \min(\lambda, \gamma(f \geq \lambda)) = \max_{A \subseteq S} \min(\gamma(A), \min_{s \in A} f(s))$$

Moreover : $\mathcal{S}_\Pi(f) = \max_{s \in S} \min(\pi(s), f(s))$; $\mathcal{S}_N(f) = \min_{s \in S} \max(\nu(\pi(s)), f(s))$.

Proposition: $\mathcal{S}_\gamma(f) = \inf_{\pi \in \mathcal{R}_*(\gamma)} \mathcal{S}_\Pi(f)$

Proof: $\mathcal{S}_\gamma(f) \leq \inf_{\pi \in \mathcal{R}_*(\gamma)} \mathcal{S}_\Pi(f)$ is obvious.

Conversely, Define the possibility measure $\Pi_f \geq \gamma$ such that $\Pi_f(f \geq \lambda) = \gamma(f \geq \lambda), \forall \lambda \in L$ and π_f the corresponding possibility distribution. Then as $\exists \pi \in \mathcal{R}_*(\gamma), \pi_f \geq \pi$, by definition,

$$\mathcal{S}_\gamma(f) = \mathcal{S}_{\Pi_f}(f) \geq \mathcal{S}_\Pi(f) \geq \inf_{\pi \in \mathcal{R}_*(\gamma)} \mathcal{S}_\Pi(f).$$

Remark Using conjugacy properties, $\mathcal{S}_\gamma(f) = \sup_{\pi \in \mathcal{R}_*(\gamma^c)} \mathcal{S}_N(f)$ as well.

n-minitivity

The complexity of a qualitative q-capacity can be assessed by the number of possibility or necessity measures needed to define it, that is the number of possibility distributions in $\mathcal{R}_*(\gamma)$ or in $\mathcal{R}_*(\gamma^c)$.

- The choice is not immaterial : they do not contain the same number of elements.
- We can find a condition under which $\gamma(A) = \max_{i=1}^n N_i(A)$ where none of the N_i are redundant.

  Proposition: $\gamma(A) = \max_{i=1}^n N_i(A)$ if and only if $\forall A_i, i = 1, \ldots n+1, \min_{i=1}^{n+1} \gamma(A_i) \leq \max_{i \neq j} \gamma(A_i \cap A_j)$
- for $n = 1$ (necessity measures) the property comes down to $\min(\gamma(A), \gamma(B)) = \gamma(A \cap B)$.
- Likewise $\gamma(A) = \min_{i=1}^n \Pi_i(A)$ if and only if $\forall A_i, i = 1, \ldots n+1, \max_{i=1}^{n+1} \gamma(A_i) \geq \min_{i \neq j} \gamma(A_i \cup A_j)$

Modal logic, possibility and necessity measures

Consider a propositional language $\mathcal{L}$ with variables $\{a, b, c...\}$ with standard connectives $\wedge, \vee, \neg, \rightarrow$ generating the set $S : p \in \mathcal{L} \iff p = a|\neg p|p \wedge q$ and $p \vee q \equiv \neg(\neg p \wedge \neg q)$.

- Define $\Box p$ as standing for $N(A) \geq \lambda > 0$, where $A = [p]$ the set of models of p. $\Box p$ corresponds to a Boolean necessity measure based on a possibility distribution that is the characteristic function of $E = \{s|\pi(s) > \nu(\lambda)\}$
- Consider a higher level propositional language $\phi \in \mathcal{L}_\Box \iff \phi = \Box p|\neg\phi|\phi \wedge \psi$.
- It is the language of Boolean possibilistic logic: $\models \Diamond p$ stands for $\Pi(A) \geq \nu(\lambda)$.
- The following KD axioms are valid
  - $(K) : \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$.
  - $(N) : \quad \Box\top$.
  - $(D) : \quad \Box p \rightarrow \Diamond p$

  and imply axiom (C) : $\Box(p \wedge q) \equiv (\Box p \wedge \Box q)$ (Boolean minitivity axiom).

Semantics of Boolean possibilistic logic

A "model" of a formula in $\phi \in \mathcal{L}_\Box$ is a nonempty subset $E \subseteq S$ of propositional models.

E is an epistemic state (a *meta-model*).

The satisfaction of MEL-formulae is defined recursively:

- $E \models \Box p$, if and only if $E \subseteq [p]$
- $E \models \neg\phi$, if and only if $E \not\models \phi$,
- $E \models \phi \wedge \psi$, if and only if $E \models \phi$ and $E \models \psi$, where ϕ, ψ are any $\mathcal{L}_\Box$-formulae.
- So, $E \models \Diamond p$ if and only if $E \cap [p] \neq \emptyset$

For any set $\Gamma \cup \{\phi\}$ of $\mathcal{L}_\Box$-formulae, ϕ is a semantic consequence of Γ, written $\Gamma \models \phi$, provided for every epistemic state E, $E \models \Gamma$ implies $E \models \phi$.

Boolean possibilistic logic (the $\mathcal{L}_\Box$-fragment of KD) is sound and complete w.r.t. this semantics

Qualitative capacities and modal logic

Using the same language, denote $\models \Box p$ as standing for $\gamma([p]) \geq \lambda > 0$.

- The following axioms are then verified :

(RE) : $\Box p \equiv \Box q$ whenever $\vdash p \equiv q$.

(RM) : $\Box p \rightarrow \Box q$, whenever $\vdash p \rightarrow q$.

(N) : $\Box \top$.

(P) : $\Diamond \top$

- This modal logic seems to be the natural logical account of qualitative capacities
- This logic is a non-regular modal logic. It is a special case of the *monotonic modal* logic EMN (Chellas), a fragment where modalities only apply to propositions, not to modal formulas. Its usual semantics is neighbourhood semantics.
- this logic does not satisfy axioms K, C nor D.

Capacity semantics of monotonic modal logics ; n-minitive case

Let n be the smallest integer for which $\gamma(A) = \max_{i=1}^{n} N_i(A)$.

- Denoting by $\Box_i p$ the statement $N_i([p]) \geq \lambda > 0$, it is clear that $\gamma([p]) \geq \lambda > 0$, i.e. $\Box p$ stands for $\vee_{i=1}^{n} \Box_i p$, where $\Box_i$ are KD modalities.
- Applying the characterisation of n-minitivity, the restriction of the modal logic EMN to the semantics in terms of n-minitive capacities is

  $$n\text{-}C :\vdash (\wedge_{i=1}^{n+1} \Box p_i) \rightarrow \vee_{i \neq j=1}^{n+1} \Box(p_i \wedge p_j)$$

  which implies that if $p_i, i = 1 \ldots, n+1$ are mutually inconsistent, then $\vdash \neg \wedge_{i=1}^{n+1} \Box p_i$ (cannot have $\gamma([p_i]) \geq \lambda > 0$ for all $i = 1 \ldots, n+1$.
- For $n = 1$ this is axiom $C : \Box(p \wedge q) \equiv (\Box p \wedge \Box q)$

Neighborhood semantics of q-capacity-based modal logic

The semantics of $EMNP + n$-C logic can be expressed

- In terms of n-tuple of epistemic states (subsets of S):
  $(E_1, \ldots, E_n) \models \Box p$ if $\exists i \in [1, n], E_i \models \Box_i)$.
- in terms of neigborhoods (non-empty subsets $\mathcal{N}$ of 2^S) :
  - $\mathcal{N} \models \Box p$ if and only if $[p] \in \mathcal{N}$
  - $\mathcal{N} \models \Diamond p$ if and only if $[\neg p] \notin \mathcal{N}$
- For a KD modality $\mathcal{N} = \{A, N(A) \geq \lambda\} = \{A | A \supseteq E\}$ for some non-empty $E \subseteq S$ (a proper filter)
- for an EMNP modality $\mathcal{N} = \{A, \gamma(A) \geq \lambda > 0\}$ ($\neq 2^S$, closed under inclusion and not empty)
- for an EMNP+n-C modality, $\mathcal{N} = \{A, \gamma(A) \geq \lambda > 0\}$ is the union of n proper filters of the form $\{A, N_i(A) \geq \lambda\} = \{A | A \supseteq E_i\}$.

Conclusion

There is a strong similarity between q-capacities and imprecise probabilities, where possibility measures replace probability measures.

- A q-capacity is both a lower possibility or an upper necessity
- A q-capacity can be viewed as a kind of belief function or a kind of plausibility function
- The equivalent (max-convex) possibility set is never empty.
- We can lay bare the minimal set of possibility measures that can reconstruct the q-capacity.
  - Note that if γ is strictly monotone : $\gamma(E) = \gamma_{\#}(E) > 0, \forall E \neq \emptyset \subseteq S$, then $\mathcal{R}_*(\gamma)$ contains $n!$ possibility distributions.
  - One can choose between upper or lower representation: It is not worth approximating N from above by a family of possibility distributions.

Perspectives

- Compare k-minitive (maxitive) q-capacities with capacities having focal sets with less than k elements in terms of representational complexity (k-maxitive in the sense of Grabish-Mesiar.
- Study the class of q-capacities induced by a small number of possibility distributions as to their potential in practical multicriteria decision problems.
- Develop the analogy with belief functions (Prade, Rico, ECSQARU 2011) : specificity, combination rules : a qualitative theory of evidence and merging unreliable testimonies?
- Develop the bridge with modal logics : generalizing possibilistic logic to q-capacities, and using it in multi source epistemic reasoning systems.
- Do it again when L is a De Morgan lattice !