Duality of aggregation operators and the explicit expression of k-negations

E. de Amo[1] M. Díaz Carrillo[2] J. Fernández Sánchez[1]

Universidad de Almería[1] Universidad de Granada[2]
Supported by Project No. MTM2011-22394, by the Ministerio de Ciencia e Innovación (Spain)

FSTA 2012

Aggregation operators

Definition

Aggregation operators are mathematical objects that have the goal of reducing a set of numbers into a unique representative (or meaningful) number. Let $\mathbb{I}^2$ be the unit square. An *aggregation operator* is defined as a function $F : \mathbb{I}^2 \longrightarrow \mathbb{I}$ that satisfies:

- (i) $F(0,0) = 0$ and $F(1,1) = 1$ (*boundary conditions*)
- (ii) $F(x_1, y_1) \leq F(x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ (*non-decreasing monotonicity*).

Aggregation operators

Definition

Aggregation operators are mathematical objects that have the goal of reducing a set of numbers into a unique representative (or meaningful) number. Let $\mathbb{I}^2$ be the unit square. An *aggregation operator* is defined as a function $F : \mathbb{I}^2 \longrightarrow \mathbb{I}$ that satisfies:

- (i) $F(0,0) = 0$ and $F(1,1) = 1$ (*boundary conditions*)
- (ii) $F(x_1, y_1) \leq F(x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ (*non-decreasing monotonicity*).

Aggregation operators appear in:

- Multidecision
- Fuzzy logic (t-norms)
- Image processing

Aggregation operators

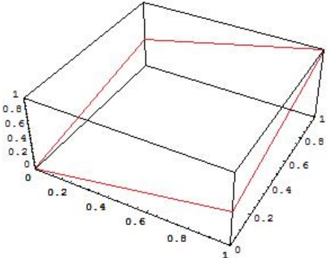
Definition

We will denote by Φ the subclass of commutative aggregation operators F that satisfy the relations:

$$F(x,0) = F(1,0)x \text{ and } F(x,1) = (1 - F(1,0))x + F(1,0),$$

for all $x \in \mathbb{I}$, with $F(1,0) \in]0,1[$.

Let us observe that t-norms and t-conorms are in Φ.



Negations and Duality

Definition

A *negation* N is defined as a non-increasing function $N : \mathbb{I} \longrightarrow \mathbb{I}$ with boundary conditions $N(0) = 1, N(1) = 0$. If N is involutive, i.e. if $N(N(x)) = x$ holds for all $x \in \mathbb{I}$, we say that N is a *strong negation*.

Negations and Duality

Definition

A *negation* N is defined as a non-increasing function $N : \mathbb{I} \to \mathbb{I}$ with boundary conditions $N(0) = 1, N(1) = 0$. If N is involutive, i.e. if $N(N(x)) = x$ holds for all $x \in \mathbb{I}$, we say that N is a *strong negation*.

Definition

Let T, S be in Φ and let N be a negation function. N is said to be a *duality function for the pair* (T, S) (or that *the pair* (T, S) *is* N*-dual*), if $N(T(x, y) = S(N(x), N(y))$ for all x, y in $\mathbb{I}$.

Target

Mayor and Torrens studied the set Φ and a duality relation for pairs of members in Φ.

Theorem ([5, Th.2])

Let F be in Φ. Given $k, k\acute{}, 0 < k, k\acute{} < 1$, there exists a unique $G_{F,k'}$ in Φ, with $G_{F,k'}(1,0) = k'$, and a unique negation function $N_{k,k'} : \mathbb{I} \rightarrow \mathbb{I}$ such that the pair $(F, G_{F,k'})$ is $N_{k,k'}$-dual.

We want to give an explicit expression for $N_{k,k'}$ and study its properties.

Target

- Properties:

Target

- Properties:
  - Derivation properties

Target

- Properties:
  - Derivation properties
  - Functional equation characterization

Target

- Properties:
  - Derivation properties
  - Functional equation characterization
  - Hausdorff dimensions

Target

- Properties:
  - Derivation properties
  - Functional equation characterization
  - Hausdorff dimensions
- $k-$negations

Representation system

Theorem

Let $k \in \,]0,1[$. If $x \in \,]0,1]$, then there is an increasing sequence of naturals $1 \leq m_0 \leq m_1 \leq \cdots \leq m_d \leq \cdots$, such that $x = \sum_{d=0}^{+\infty} (1-k)^d \, k^{m_d}$. Besides, the above expansion is unique but it would be finite or stationary (i.e., $m_d = m_j$ if $d \geq j$).

Definition

Definition

For each pair $k, k' \in \,]0,1[\, \setminus \{\frac{1}{2}\}$, let us define the function $f_{k,k'} : \mathbb{I} \longrightarrow \mathbb{I}$, given in the following way: each x with non-stationary infinite expansion (that is, there exist $1 \leq t_0 < t_1 < \cdots < t_d < \cdots$) such that

$$
\begin{aligned}
x \;=\; & k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} \\
& + k^{t_1}(1-k)^{s_0+1} + \cdots + k^{t_1}(1-k)^{s_1} + \cdots \\
& + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d} + \cdots
\end{aligned}
$$

Definition

is mapped to

$$
\begin{aligned}
f_{k,k'}(x) \quad = \quad & k' + k'\left(1-k'\right) + \cdots + k'\left(1-k'\right)^{t_0-2} + \\
& k'^{s_0+2}\left(1-k'\right)^{t_0-1} + \cdots + k'^{s_0+2}\left(1-k'\right)^{t_1-2} + \\
& k'^{s_1+2}\left(1-k'\right)^{t_1-1} + \cdots + k'^{s_1+2}\left(1-k'\right)^{t_2-2} + \cdots \\
& k'^{s_{d-1}+2}\left(1-k'\right)^{t_{d-1}-1} + \cdots + \\
& k'^{s_{d-1}+2}\left(1-k'\right)^{t_d-2} + \cdots
\end{aligned}
$$

Definition

If $t_0 := 1$, then $k + k(1-k) + \cdots + k(1-k)^{t_0-2}$ does not exist. In the stationary case, that is, if x has finite expansion:

$$\begin{aligned} x &= k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + \\ &\quad k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d}, \end{aligned}$$

then

$$\begin{aligned} f_{k,k'}(x) &:= k' + k'(1-k') + \cdots + k'(1-k')^{t_0-2} + \cdots + \\ &\quad k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + \\ &\quad k'^{s_{d-1}+2}(1-k')^{t_d-2} + k'^{s_d+1}(1-k')^{t_d-1}. \end{aligned}$$

Example

$$
\begin{aligned}
x \quad = \quad & k^2 + k^2(1-k) + \cdots + k^2(1-k)^5 + \\
& k^4(1-k)^6 + \cdots + k^4(1-k)^{11} + \\
& k^7(1-k)^{12} + \cdots + k^7(1-k)^{16} + \\
& k^{11}(1-k)^{17} + \cdots + k^{11}(1-k)^{22} + \\
& k^{25}(1-k)^{23} + \cdots + k^{25}(1-k)^{30} + \cdots
\end{aligned}
$$

Then the values for t_i and s_i are given by:

$$
\begin{array}{ll}
t_0 = 2 & s_0 = 5 \\
t_1 = 4 & s_1 = 11 \\
t_2 = 7 & s_2 = 16 \\
t_3 = 11 & s_3 = 22 \\
t_4 = 25 & s_4 = 30 \\
\cdots & \cdots
\end{array}
$$

Example

the first terms for the series expansion of $f_{k,k'}(x)$ are:

$$\begin{aligned} f_{k,k'}(x) &= k' + k'^7\left(1-k'\right) + k'^7\left(1-k'\right)^2 + \\ &\quad k'^{13}\left(1-k'\right)^3 + k'^{13}\left(1-k'\right)^4 + k'^{13}\left(1-k'\right)^5 + \\ &\quad k'^{18}\left(1-k'\right)^6 + \cdots + k'^{18}\left(1-k'\right)^9 + \\ &\quad k'^{24}\left(1-k'\right)^{10} + \cdots + k'^{24}\left(1-k'\right)^{23} + \cdots \end{aligned}$$

Graphs

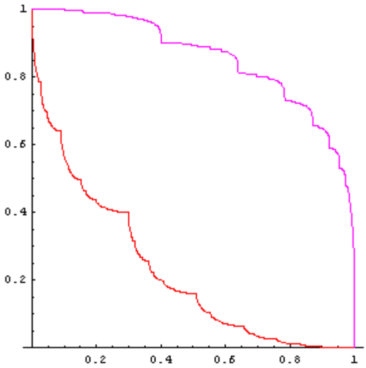


Figure: The Graphs of $N_{.3,.4}$ and $N_{.4,.9}$

Properties

Theorem

The functions $N_{k,k'}$ and $f_{k,k'}$ coincide.

Theorem

If $k \neq 1 - k'$, then there exists a set of measure 1 in which the derivative of $N_{k,k'}$ vanishes.

Properties

Theorem

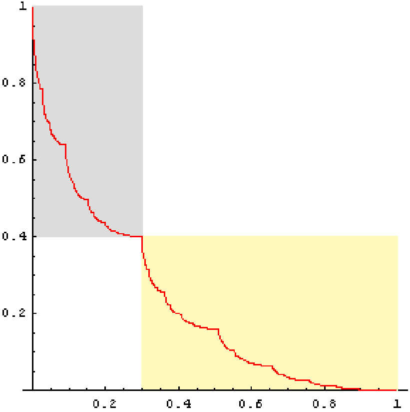
If $k \neq 1 - k'$, then $N_{k,k'}$ does not admit a non-zero derivative at any $x \in \mathbb{I}$.

Theorem

$N_{k,k'}$ is the unique bounded solution of the system of functional equations

$$\begin{cases} f(kx) = k' + (1-k') f(x) \\ f(k + (1-k)x) = k' f(x). \end{cases}$$

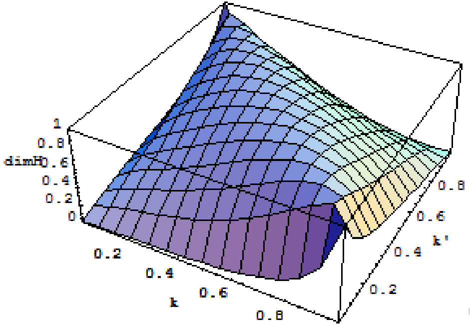
Properties



Properties

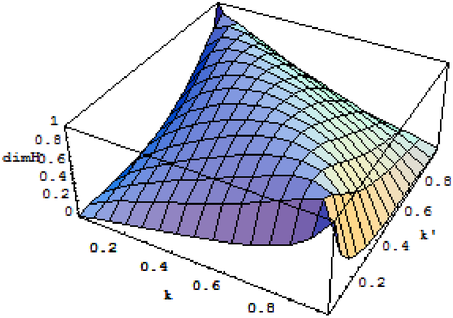
Theorem

If $k' \neq 1-k$, then the function $N_{k,k'}$ applies a set of λ-measure 0 onto a set of λ-measure 1. The Hausdorff dimension of the first set is $\dfrac{\ln\left[k'^{k'}(1-k')^{1-k'}\right]}{\ln\left[k^{1-k'}(1-k)^{k'}\right]}$.



Theorem

If $k' \neq 1 - k$, then $N_{k,k'}$ applies a set of λ-measure 1 onto a set of λ-measure 0 whose Hausdorff dimension is $\dfrac{\ln\left[k^k(1-k)^{1-k}\right]}{\ln\left[k'^{1-k}(1-k')^k\right]}$.



Negations

Theorem

For each $k \in \,]0,1[\,$, let us consider the k-negation function $N_k : \mathbb{I} \longrightarrow \mathbb{I}$ (under the above expression). Then,

i) N_k is continuous.

ii) For each $k \in \,]0,1[\,\backslash 1/2$, there is a set of λ-measure 1 in which N_k vanishes.

iii) For each $k \in \,]0,1[\,\backslash 1/2$, N_k does not admit non-zero derivatives.

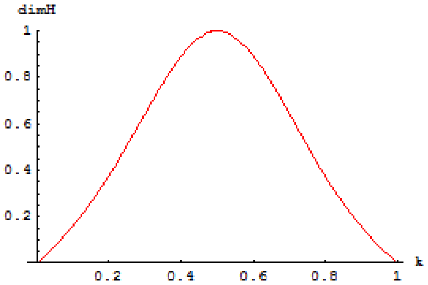
iv) N_k is the unique solution for the system of functional equations given by

$$\begin{cases} f(kx) = k + (1-k) f(x) \\ f(k + (1-k)x) = kf(x). \end{cases}$$

Theorem

v) If $k \neq 0.5$, N_k maps a set of λ-measure 0 with Hausdorff dimension $\frac{\ln\left[k^k(1-k)^{1-k}\right]}{\ln\left[k^{1-k}(1-k)^k\right]}$, onto a set of λ-measure 1.

vi) If $k \neq 0.5$, N_k maps a set of λ-measure 1 onto a set of λ-measure 0 with Hausdorff dimension $\frac{\ln\left[k^k(1-k)^{1-k}\right]}{\ln\left[k^{1-k}(1-k)^k\right]}$



References

- Alsina, C., Frank, M.J. and Schweizer, B.: Associative Functions: Triangular Norms and Copulas, World Scientific, Singapur, 2006
- de Amo, E. and Fernández Sánchez, J.: *A Generalised Dyadic Representation System*, Int. J. Pure Appl. Math. **52** (1) (2009) 49-66
- de Amo, E., Díaz Carrillo, M. and Fernández Sánchez, J.: *On duality of aggregation operators and k-negations*, Fuzzy Sets and Systems **181** (2011) 14-27
- Mayor, G. and Calvo, T.: *Fractal Negations*, Mathware Soft Comput. **3** (1994) 277-283
- Mayor, G. and Torrens, J.: *Duality for a class of binary operations on* $[0,1]$, Fuzzy Sets and Systems **47** (1992) 77-80

Thank you for your attention!