On multi-objective linear programming approach for solving fuzzy matrix games

Diāna Dance

Department of Mathematics, University of Latvia

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In this talk we deal with non cooperative two-person games with fuzzy pay-offs. Namely, we consider matrix games where each component of the pay-off matrix is a fuzzy number. We describe the formal definition of the value of a fuzzy pay-off matrix game and develop a fuzzy programming method to find it by solving the corresponding bilevel linear programming problem. To realize fuzzy programming on two levels we apply specially designed aggregation of objectives.

Matrix games

Matrix games are zero - sum two - person games.
A is called the pay-off matrix: if Player I chooses i^{th} $(i = \overline{1, m})$ strategy and Player II chooses j^{th} $(j = \overline{1, n})$ strategy then a_{ij} is amount paid by Player II to Player I:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Lower value of the game: $v = \max_{i=\overline{1,m}} \min_{j=\overline{1,n}} a_{ij}$.
Upper value of the game: $w = \min_{j=\overline{1,n}} \max_{i=\overline{1,m}} a_{ij}$.
If $v = w$ then it is called value of the game.

Expected pay-off function

$$E_A(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j,$$

when Player I chooses mixed strategy x and Player II - y:

$$S^m = \{x = (x_1, x_2, \ldots, x_m) : \quad \sum_{i=1}^{m} x_i = 1, \quad x_i \in [0, 1]\},$$

$$S^n = \{y = (y_1, y_2, \ldots, y_n) : \quad \sum_{j=1}^{n} y_j = 1, \quad y_j \in [0, 1]\}.$$

The function E_A has a seddle point (x^*, y^*) and x^*, y^* are optimal strategies for Player I and Player II.
Value of the game $E_A(x^*, y^*)$.

Linear programming and matrix game equivalence

Player I:

$v \to \max,$

$$\sum_{i=1}^{m} a_{ij} x_i \geqslant v \quad (j = \overline{1, n}),$$

$$\sum_{i=1}^{m} x_i = 1,$$

$$x_i \geqslant 0 \quad (i = \overline{1, m}).$$

Player II:

$w \to \min,$

$$\sum_{j=1}^{n} a_{ij} y_j \leqslant w \quad (i = \overline{1, m}),$$

$$\sum_{j=1}^{n} y_j = 1,$$

$$y_j \geqslant 0 \quad (j = \overline{1, n}).$$

Solutions of the problems: x^*, y^* and v^*, w^*.

Fuzzy matrix games

Pay-off matrix:

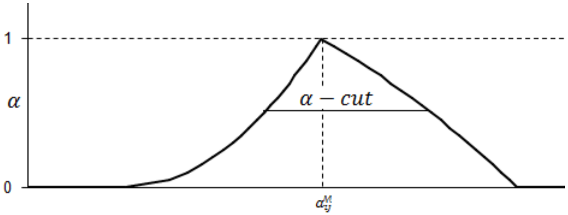
$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \dots & \dots & \dots & \dots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \dots & \tilde{a}_{mn} \end{pmatrix}$$

where $\tilde{a}_{ij}$ $\left(i = \overline{1, m}; j = \overline{1, n}\right)$ are fuzzy numbers.

Fuzzy numbers (FN)

Fuzzy number is a function $\tilde{a}_{ij} : R \rightarrow [0,1]$ if:

- there exists the unique point a_{ij}^M such that $\tilde{a}_{ij}(a_{ij}^M) = 1$;
- α - cuts $\tilde{a}_{ij}|_\alpha$ are closed for all $\alpha \in [0,1]$.



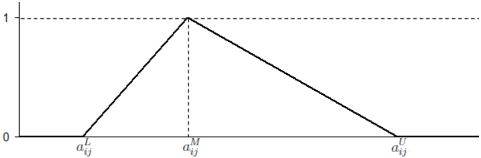
We denote: $\tilde{a}_{ij}^L = \tilde{a}_{ij}|_{]-\infty;a_{ij}^M]}$, $\tilde{a}_{ij}^U = \tilde{a}_{ij}|_{[a_{ij}^M;\infty[}$.

α - cut for $\tilde{a}_{ij}$ will be $[a_{ij}^L|_\alpha; a_{ij}^U|_\alpha]$.

Triangular fuzzy numbers (TFN)

Triangular fuzzy number:

$$\tilde{a}_{ij} = (a_{ij}^L, a_{ij}^M, a_{ij}^U), \text{ where } a_{ij}^L < a_{ij}^M < a_{ij}^U.$$



Membership function:

$$\tilde{a}_{ij}(a_{ij}) = \begin{cases} 0, & \text{if } a_{ij} \leqslant a_{ij}^L, a_{ij} \geqslant a_{ij}^U, \\ \dfrac{a_{ij} - a_{ij}^L}{a_{ij}^M - a_{ij}^L}, & \text{if } a_{ij}^L \leqslant a_{ij} \leqslant a_{ij}^M, \\ \dfrac{a_{ij} - a_{ij}^U}{a_{ij}^M - a_{ij}^U}, & \text{if } a_{ij}^M \leqslant a_{ij} \leqslant a_{ij}^U. \end{cases}$$

Multi-objective programming approach: Li's model

Player 1

$(v^L, v^M, v^U) \to \max$

$$\sum_{i=1}^{m} a_{ij}^L x_i \geqslant v^L, \quad (j = 1, .., n)$$

$$\sum_{i=1}^{m} a_{ij}^M x_i \geqslant v^M, \quad (j = 1, .., n)$$

$$\sum_{i=1}^{m} a_{ij}^U x_i \geqslant v^U, \quad (j = 1, .., n)$$

$$\sum_{i=1}^{m} x_i = 1, x_i \geqslant 0$$

Player 2

$(w^L, w^M, w^U) \to \min$

$$\sum_{j=1}^{n} a_{ij}^L y_j \leqslant w^L, \quad (i = 1, .., m)$$

$$\sum_{j=1}^{n} a_{ij}^M y_j \leqslant w^M, \quad (i = 1, .., m)$$

$$\sum_{j=1}^{n} a_{ij}^U y_j \leqslant w^U, \quad (i = 1, .., m)$$

$$\sum_{j=1}^{n} y_j = 1, y_j \geqslant 0$$

Reasonable solution of fuzzy matrix games

Ordering of TFN's

$\tilde{t} = (t^L, t^M, t^U)$, $\tilde{\tau} = (\tau^L, \tau^M, \tau^U)$ are TFN's. Then $\tilde{t} \leqslant \tilde{\tau}$ if $t^L \leqslant \tau^L$, $t^M \leqslant \tau^M$ and $t^U \leqslant \tau^U$.

Definition

Let $\tilde{v} = (v^L, v^M, v^U)$ and $\tilde{w} = (w^L, w^M, w^U)$ be TFN's. Then $(\tilde{v}, \tilde{w})$ is called a reasonable solution of the fuzzy matrix game if there exist $\overline{x} \in S^m$, $\overline{y} \in S^n$ such that

- $E_{\tilde{A}}(\overline{x}, y) \geqslant \tilde{v}$ for all $y \in S^n$;
- $E_{\tilde{A}}(x, \overline{y}) \leqslant \tilde{w}$ for all $x \in S^m$.

If $(\tilde{v}, \tilde{w})$ is a reasonable solution of fuzzy game then $\tilde{v}$ (respectively $\tilde{w}$) is called the reasonable value of Player I (Player II).

Solutions of fuzzy matrix games

V (respectively W) is the set of all reasonable values $\tilde{\mathrm{v}}$ (respectively $\tilde{\mathrm{w}}$) for Player I (Player II).

Definition

An element $(\tilde{\mathrm{v}}_* = (\mathrm{v}_*^L, \mathrm{v}_*^M, \mathrm{v}_*^U), \tilde{\mathrm{w}}_* = (\mathrm{w}_*^L, \mathrm{w}_*^M, \mathrm{w}_*^U)) \in \mathrm{V} \times \mathrm{W}$ is called a solution of the fuzzy game if

- there does not exist any $\tilde{\mathrm{v}} = (\mathrm{v}^L, \mathrm{v}^M, \mathrm{v}^U) \in \mathrm{V}$ such that $(\mathrm{v}^L, \mathrm{v}^M, \mathrm{v}^U) \geqslant (\mathrm{v}_*^L, \mathrm{v}_*^M, \mathrm{v}_*^U)$;
- there does not exist any $\tilde{\mathrm{w}} = (\mathrm{w}^L, \mathrm{w}^M, \mathrm{w}^U) \in \mathrm{W}$ such that $(\mathrm{w}^L, \mathrm{w}^M, \mathrm{w}^U) \leqslant (\mathrm{w}_*^L, \mathrm{w}_*^M, \mathrm{w}_*^U)$.

Step 1

$$v^M \to \max_{x \in D}$$

Solution: v_*^M and x^*

Step 2

$$v^L, v^U \to \max$$

$$\sum_{i=1}^{m} a_{ij}^L x_i^* \geqslant v^L, \quad (j = 1, .., n)$$

$$\sum_{i=1}^{m} a_{ij}^U x_i^* \geqslant v^U, \quad (j = 1, .., n)$$

Solution of the problem is: (v_*^L, v_*^M, v_*^U).

Numerical example 1

$$\tilde{A} = \begin{pmatrix} \tilde{20} & \tilde{5} \\ \tilde{10} & \tilde{20} \end{pmatrix},$$

where $\tilde{20} = (10, 20, 30)$, $\tilde{5} = (1, 5, 20)$, $\tilde{10} = (9, 10, 15)$, $\tilde{20} = (2, 20, 30)$ are TFN's.
If we solve individual problems of

$$(v^L, v^M, v^U) \to \max_{x \in D_1}$$

then

$$x_*^M = (0.4, 0.6), x_*^L = (0, 1), x_*^U = (0.6, 0.4)$$

Bi-level linear programming

Player 1

$$P_1^1 : v^M \to \max$$
$$P_2^1 : v^L \to \max$$
$$v^U \to \max$$

$$\sum_{i=1}^{m} a_{ij}^L x_i \geqslant v^L \quad (j = \overline{1,n}),$$

$$\sum_{i=1}^{m} a_{ij}^M x_i \geqslant v^M \quad (j = \overline{1,n}),$$

$$\sum_{i=1}^{m} a_{ij}^U x_i \geqslant v^U \quad (j = \overline{1,n}),$$

$$\sum_{i=1}^{m} x_i = 1, x_i \geqslant 0.$$

Player 2

$$P_1^2 : w^M \to \min$$
$$P_2^2 : w^L \to \min$$
$$w^U \to \min$$

$$\sum_{j=1}^{n} a_{ij}^L y_j \leqslant w^L \quad (i = \overline{1,m}),$$

$$\sum_{j=1}^{n} a_{ij}^M y_j \leqslant w^M \quad (i = \overline{1,m}),$$

$$\sum_{j=1}^{n} a_{ij}^U y_j \leqslant w^U \quad (i = \overline{1,m}),$$

$$\sum_{j=1}^{n} y_j = 1, y_j \geqslant 0.$$

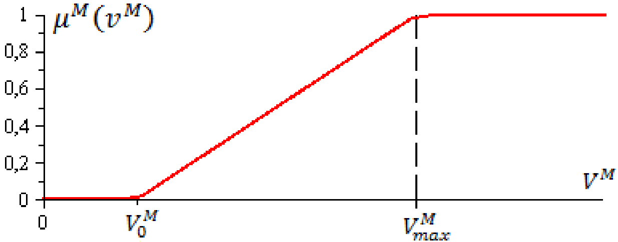
Individual solutions

- $v^M \to \max_{D_1}$
  Solutions:
  $v^M_{max} = v^M(x^M)$, where $x^M = (x^M_1, ..., x^M_m)$
- $v^L \to \max_{D_1}$
  Solutions:
  $v^L_{max} = v^L(x^L)$, where $x^L = (x^L_1, ..., x^L_m)$
- $v^U \to \max_{D_1}$
  Solutions:
  $v^U_{max} = v^U(x^U)$, where $x^U = (x^U_1, ..., x^U_m)$

Membership function

$$\mu^{\mathrm{M}}(\mathrm{v}^{\mathrm{M}}) = \begin{cases} 0, & \text{if } \mathrm{v}^{\mathrm{M}} \leqslant \mathrm{v}_0^{\mathrm{M}}, \\ \dfrac{\mathrm{v}^{\mathrm{M}} - \mathrm{v}_0^{\mathrm{M}}}{\mathrm{v}_{\mathrm{max}}^{\mathrm{M}} - \mathrm{v}_0^{\mathrm{M}}}, & \text{if } \mathrm{v}_0^{\mathrm{M}} \leqslant \mathrm{v}^{\mathrm{M}} \leqslant \mathrm{v}_{\mathrm{max}}^{\mathrm{M}}, \\ 1, & \text{if } \mathrm{v}^{\mathrm{M}} \geqslant \mathrm{v}_{\mathrm{max}}^{\mathrm{M}}. \end{cases}$$

where $\mathrm{v}_0^{\mathrm{M}} = \min(\mathrm{v}^{\mathrm{M}}(\mathrm{x}^{\mathrm{L}}), \mathrm{v}^{\mathrm{M}}(\mathrm{x}^{\mathrm{U}}))$.



To maximize membership functions:

$$\mu^{\mathrm{M}}(\mathrm{v}^{\mathrm{M}}), \mu^{\mathrm{L}}(\mathrm{v}^{\mathrm{L}}), \mu^{\mathrm{U}}(\mathrm{v}^{\mathrm{U}}) \longrightarrow \max_{\mathrm{x}\in \mathrm{D}_1},$$

we maximize the smallest extreme degree of achievement among all functions:

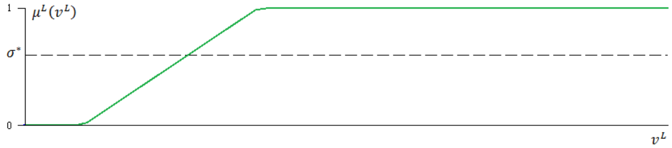
$$\min\{\mu^{\mathrm{M}}(\mathrm{v}^{\mathrm{M}}), \mu^{\mathrm{L}}(\mathrm{v}^{\mathrm{L}}), \mu^{\mathrm{U}}(\mathrm{v}^{\mathrm{U}})\} \longrightarrow \max_{\mathrm{x}\in \mathrm{D}_1}$$

We denote $\sigma : \langle \mu^{\mathrm{M}}(\mathrm{v}^{\mathrm{M}}) \geqslant \sigma, \mu^{\mathrm{L}}(\mathrm{v}^{\mathrm{L}}) \geqslant \sigma, \mu^{\mathrm{U}}(\mathrm{v}^{\mathrm{U}}) \geqslant \sigma \rangle$ and write problem in this form:

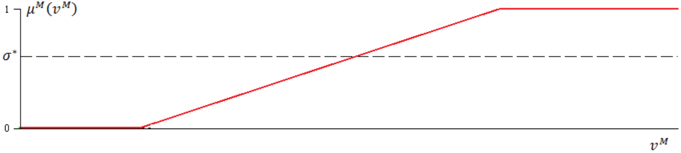
$$\sigma \longrightarrow \max_{\mathrm{x},\sigma}$$

$$\begin{cases} \mu^{\mathrm{M}}(\mathrm{v}^{\mathrm{M}}) \geqslant \sigma \\ \mu^{\mathrm{L}}(\mathrm{v}^{\mathrm{L}}) \geqslant \sigma \\ \mu^{\mathrm{U}}(\mathrm{v}^{\mathrm{U}}) \geqslant \sigma \\ \mathrm{x} \in \mathrm{D}_1 \end{cases}$$

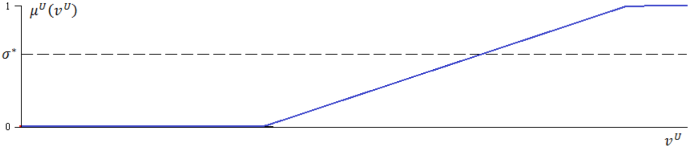
- membership function $\mu^{\mathrm{L}}(\mathrm{v}^{\mathrm{L}})$

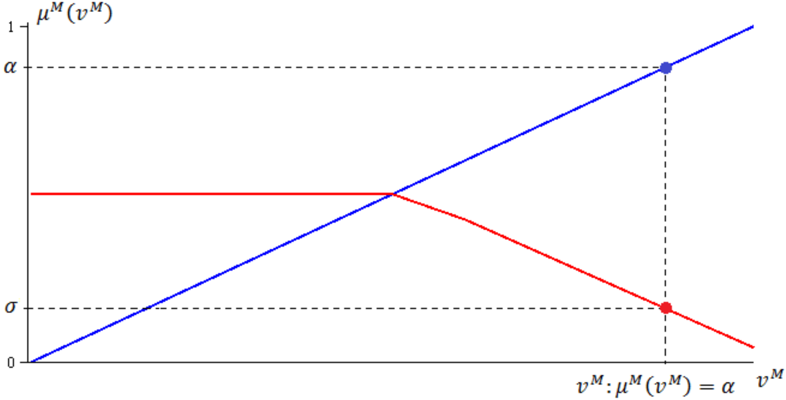


- membership function $\mu^{\mathrm{M}}(\mathrm{v}^{\mathrm{M}})$



- membership function $\mu^{\mathrm{U}}(\mathrm{v}^{\mathrm{U}})$





$$\sigma = \max_{x:\mu^M(v^M)=\alpha} \min\{\mu^L(v^L), \mu^U(v^U)\}$$

This graph is obtained by solving the following problems:

$$\sigma \longrightarrow \max_{x,\sigma}$$

$$\begin{cases} \mu^{M}(v^{M}) = \alpha \\ \mu^{L}(v^{L}) \geqslant \sigma \\ \mu^{U}(v^{U}) \geqslant \sigma \\ x \in D_1 \end{cases}$$

Numerical example 2

Numerical example (Li and Yang, Campos)

$$\tilde{A} = \begin{pmatrix} \tilde{180} & \tilde{90} \\ \tilde{156} & \tilde{180} \end{pmatrix},$$

where $\tilde{180} = (175, 180, 190)$, $\tilde{156} = (150, 156, 158)$, $\tilde{90} = (80, 90, 100)$ are TFN's.

Individual solutions:

- $v^M \to \max_{D_1}$
  Solutions: $v^M_{max} = 161.052$, $x^M = (0.789, 0.210)$
- $v^L \to \max_{D_1}$
  Solutions: $v^L_{max} = 155.208$, $x^L = (0.791, 0.208)$
- $v^U \to \max_{D_1}$
  Solutions: $v^U_{max} = 166.393$, $x^U = (0.737, 0.262)$

$$v_0^M = 156.393, v_0^L = 150.081, v_0^U = 164.666$$

Membership functions

$$\mu^{\mathrm{M}}(\mathrm{v}^{\mathrm{M}}) = \begin{cases} 0, & \text{if } \mathrm{v}^{\mathrm{M}} \leqslant 156.393, \\ \dfrac{\mathrm{v}^{\mathrm{M}} - 156.393}{161.052 - 156.393}, & \text{if } 156.393 \leqslant \mathrm{v}^{\mathrm{M}} \leqslant 161.052, \\ 1, & \text{if } \mathrm{v}^{\mathrm{M}} \geqslant 161.052. \end{cases}$$

$$\mu^{\mathrm{L}}(\mathrm{v}^{\mathrm{L}}) = \begin{cases} 0, & \text{if } \mathrm{v}^{\mathrm{L}} \leqslant 150.081, \\ \dfrac{\mathrm{v}^{\mathrm{L}} - 150.081}{155.208 - 150.081}, & \text{if } 150.081 \leqslant \mathrm{v}^{\mathrm{L}} \leqslant 155.208, \\ 1, & \text{if } \mathrm{v}^{\mathrm{L}} \geqslant 155.208. \end{cases}$$

$$\mu^{\mathrm{U}}(\mathrm{v}^{\mathrm{U}}) = \begin{cases} 0, & \text{if } \mathrm{v}^{\mathrm{U}} \leqslant 164.666, \\ \dfrac{\mathrm{v}^{\mathrm{U}} - 164.666}{166.393 - 164.666}, & \text{if } 164.666 \leqslant \mathrm{v}^{\mathrm{U}} \leqslant 166.393, \\ 1, & \text{if } \mathrm{v}^{\mathrm{U}} \geqslant 166.393. \end{cases}$$

$$\sigma \longrightarrow \max_{x,\sigma}$$

$$\begin{cases} \dfrac{v^{M} - 156.393}{161.052 - 156.393} \geqslant \sigma \\ \dfrac{v^{L} - 150.081}{155.208 - 150.081} \geqslant \sigma \\ \dfrac{v^{U} - 164.666}{166.393 - 164.666} \geqslant \sigma \\ x \in D_1 \end{cases}$$

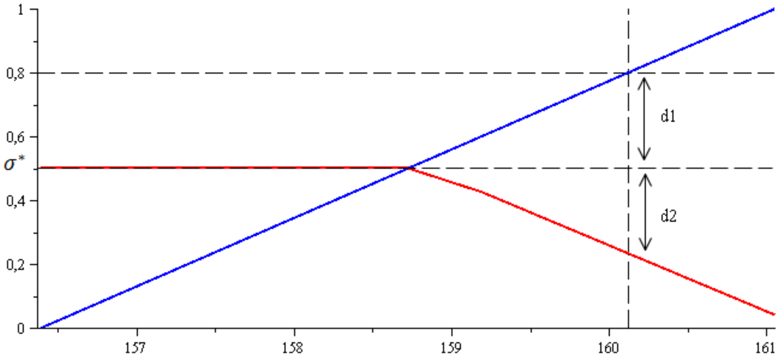
Solution:

$$\sigma^* = 0.50000001796808$$

$$\sigma \longrightarrow \max_{x,\sigma}$$

$$\begin{cases} \dfrac{v^M - 156.393}{161.052 - 156.393} = \alpha \\ \dfrac{v^L - 150.081}{155.208 - 150.081} \geqslant \sigma \\ \dfrac{v^U - 164.666}{166.393 - 164.666} \geqslant \sigma \\ x \in D_1 \end{cases}$$

α	σ	v^M
0	0.50000017968089905	156.3934
0.1	0.50000017968088351	156.8593
0.2	0.50000017968085908	157.3252
0.3	0.50000017968091681	157.7911
0.4	0.50000017968088573	158.2571
0.5	0.50000017968092902	158.7230
0.6	0.424383778061159233	159.1889
0.7	0.328447736942724334	159.6548
0.8	0.232511695824289433	160.1207
0.9	0.136575654705872296	160.5867
1	0.0406396135874516064	161.0526

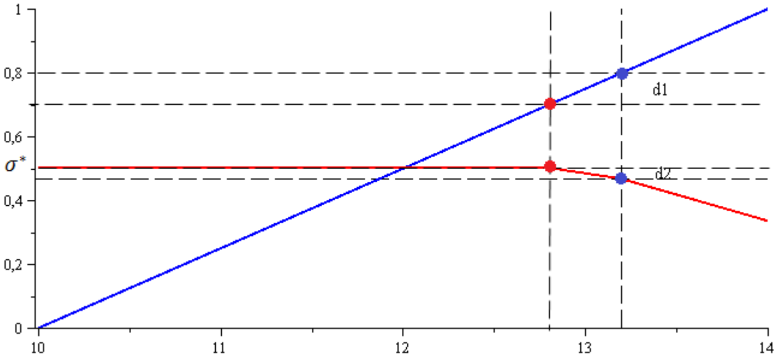


If we choose level $\delta = 0.8$ then
$x_{**} = (0.78, 0.22)$
$\mu^{M}(v^{M}_{**}) = 0.8$, $\mu^{L}(v^{L}_{**}) = 0.232511701$, $\mu^{U}(v^{U}_{**}) = 0.232511687$.

Numerical example 1

$$\tilde{A} = \begin{pmatrix} \tilde{20} & \tilde{5} \\ \tilde{10} & \tilde{20} \end{pmatrix},$$

where $\tilde{20} = (10, 20, 30)$, $\tilde{5} = (1, 5, 20)$, $\tilde{10} = (9, 10, 15)$, $\tilde{20} = (2, 20, 30)$ are TFN's.



$\sigma^* = 0.50000000017500$ If we choose level $\delta = 0.8$ then
$\mathrm{x}_{**} = (0.32, 0.68)$
$\mu^{\mathrm{M}}(\mathrm{v}^{\mathrm{M}}_{**}) = 0.8,\ \mu^{\mathrm{L}}(\mathrm{v}^{\mathrm{L}}_{**}) = 0.46666667,\ \mu^{\mathrm{U}}(\mathrm{v}^{\mathrm{U}}_{**}) = 0.46666667.$

Multi-level approach

TFN's

(Elements of the pay-off matrix are triangular fuzzy numbers)

$\Longrightarrow$

FN's

(Elements of the pay-off matrix are fuzzy numbers)

BLLP

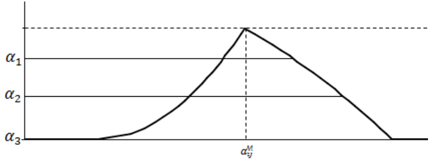
(Bi-level linear programming)

$\Longrightarrow$

MLLP

(Multi-level linear programming)

Multi-level linear programming



$$P_1^1 : v^M \to \max$$

$$P_2^1 : v^L|_{\alpha_1} \to \max$$
$$v^U|_{\alpha_1} \to \max$$

...

$$P_k^1 : v^L|_{\alpha_k} \to \max$$
$$v^U|_{\alpha_k} \to \max$$

$$\sum_{i=1}^{m} a_{ij}^M x_i \geqslant v^M \quad (j = \overline{1, n}),$$

$$\sum_{i=1}^{m} a_{ij}^L|_{\alpha_1} x_i \geqslant v^L|_{\alpha_1} \quad (j = \overline{1, n}),$$

$$\sum_{i=1}^{m} a_{ij}^U|_{\alpha_1} x_i \geqslant v^U|_{\alpha_1} \quad (j = \overline{1, n}),$$

...

$$\sum_{i=1}^{m} a_{ij}^L|_{\alpha_k} x_i \geqslant v^L|_{\alpha_1} \quad (j = \overline{1, n}),$$

$$\sum_{i=1}^{m} a_{ij}^U|_{\alpha_k} x_i \geqslant v^U|_{\alpha_1} \quad (j = \overline{1, n}),$$

$$\sum_{i=1}^{m} x_i = 1, x_i \geqslant 0.$$

Thank you for your attention!