Eigen Fuzzy Sets Equations and Related Inequalities

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Introduction

Eigen fuzzy set equations

first studied by *Sanchez (FSS, 1978; JMAA, 1981)* – used in medical research

later they were studied in many papers and found a much wider field of application

Wagenknecht, Hartmann (FSS, 1986)

construction of fuzzy eigen solutions in a given region (interval)

Jimenez, Montes, Šešelja, Tepavčević (CAMWA, 2011)

Our approach

we study both equations and inequalities

we study systems of equations (inequalities), not only single equations (inequalities)

we use more general structure of truth values (complete residuated lattice)

- ≫ differences between systems of inequalities and systems of equations
- ≫ differences between systems of equations and single equations
- ≫ connections with closures and openings w.r.t. fuzzy relations (fuzzy quasi-orders)
- ≫ algorithms for computing the greatest and the least solutions

Fuzzy sets

Structure of truth values

complete residuated lattice: $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$

negation – unary operation $\neg$ defined by $\neg x = x \to 0$

double negation property: $x = \neg\neg x$, for all $x \in L$

complete residuated lattices **do not have** this property in general

MV-algebras satisfy the double negation property

Fuzzy sets

$L^A = \{f \mid f : A \to L\}$ – the set of all fuzzy subsets of a set A over $\mathcal{L}$

$\mathcal{F}(A) = (L^A, \vee, \wedge, \emptyset, A)$ – the complete lattice of fuzzy subsets of A

complement $f^{\neg}$ of a fuzzy subset f: $f^{\neg}(x) = \neg f(x)$, for each $x \in L$

scalar multiplication λf $(= f\lambda)$ of $f \in L^A$ and $\lambda \in L$: $\lambda f(a) = \lambda \otimes f(a)$, for each $a \in A$.

$f \in L^A$ is a *linear combination* of $f_i \in L^A$ $(i \in I)$ if there are $\lambda_i \in L$ $(i \in I)$ such that

$$f = \bigvee_{i \in I} \lambda_i f_i.$$

Fuzzy sets (cont.)

Closure and opening systems

closure system in $\mathcal{F}(A)$ – a subset of L^A containing A, closed under arbitrary meets

opening system in $\mathcal{F}(A)$ – a subset of L^A containing $\emptyset$, closed under arbitrary joins

$(P, \leqslant)$ – partially ordered set. A function $\phi : P \to P$ is

- *isotone* if $x \leqslant y$ implies $\phi(x) \leqslant \phi(y)$, for all $x, y \in P$;
- *extensive* if $x \leqslant \phi(x)$, for each $x \in P$;
- *intensive* if $\phi(x) \leqslant x$, for each $x \in P$;
- *idempotent* if $\phi(\phi(x)) = \phi(x)$, for each $x \in P$.

closure operator on $(P, \leqslant)$ – *isotone*, *extensive* and *idempotent* function

if $\phi(x) = x$, then x is a ***ϕ-closed element***

opening operator on $(P, \leqslant)$ – *isotone*, *intensive* and *idempotent* function

if $\phi(x) = x$, then x is a ***ϕ-open element***

closure system in $\mathcal{F}(A)$ $\equiv$ set of all closed elements of some **closure operator** on $\mathcal{F}(A)$

opening system in $\mathcal{F}(A)$ $\equiv$ set of all open elements of some **opening operator** on $\mathcal{F}(A)$

Fuzzy sets (cont.)

Closure and opening systems (cont.)

$\mathcal{C} \subseteq L^A$ – closure system in $\mathcal{F}(A)$, $\mathcal{X} = \{f_i\}_{i\in I} \subseteq L^A$

$\mathcal{C}$-closure of $\mathcal{X}$:

$$C_{\mathcal{X}} = \bigwedge \{g \in \mathcal{C} \mid f_i \leqslant g, \text{ for every } i \in I\} \in \mathcal{C}$$

– the least element of $\mathcal{C}$ containing all members of $\mathcal{X}$.

principal element of $\mathcal{C}$ generated by an element $a \in A$:

$$C_a = \bigwedge \{f \in \mathcal{C} \mid f(a) = 1\} \in \mathcal{C}$$

– the $\mathcal{C}$-closure of the crisp subset $\{a\}$ of A

principal part of $\mathcal{C}$: $\mathcal{P}(\mathcal{C}) = \{C_a \mid a \in A\}$

$\mathcal{C} \subseteq L^A$ – opening system in $\mathcal{F}(A)$, $\mathcal{X} = \{f_i\}_{i\in I} \subseteq L^A$

$\mathcal{C}$-opening of $\mathcal{X}$:

$$C_{\mathcal{X}} = \bigvee \{g \in \mathcal{C} \mid g \leqslant f_i, \text{ for every } i \in I\} \in \mathcal{C}$$

– the greatest element of $\mathcal{C}$ contained in all members of $\mathcal{X}$

Fuzzy relations

Fuzzy relations

$L^{A\times A} = \{f \mid f : A\times A \to L\}$ – the set of all fuzzy relations on a set A over $\mathcal{L}$

$\mathcal{R}(A) = (L^{A\times A}, \vee, \wedge, \emptyset, A\times A)$ – the complete lattice of fuzzy relations on A

$R^{-1} \in \mathcal{R}(B, A)$ – *converse* (*inverse, transpose*)of a fuzzy relation $R \in \mathcal{R}(A, B)$

$$R^{-1}(b,a) \stackrel{def}{=} R(a,b) \qquad (a\in A, b\in B)$$

Compositions

$R\circ S$ – *composition of fuzzy relations* $R, S \in L^{A\times A}$:

$$(R\circ S)(a,c) = \bigvee_{b\in A} R(a,b)\otimes S(b,c) \qquad (a,c\in A)$$

$f\circ R, R\circ f$ – *compositions of a fuzzy relation* $R\in L^{A\times A}$ *and a fuzzy set* $f\in L^A$:

$$(f\circ R)(a) = \bigvee_{b\in A} f(b)\otimes R(b,a), \qquad (R\circ f)(a) = \bigvee_{b\in A} R(a,b)\otimes f(b), \qquad (a\in A)$$

n-th power R^n of a fuzzy relation R: $R^0 = \Delta_A, R^1 = R, R^{n+1} = R^n\circ R$ $(n\in\mathbb{N}^0)$.

Fuzzy quasi-orders, fuzzy equivalences

Fuzzy quasi-orders, fuzzy equivalences

$R \in L^{A\times A}$ – fuzzy relation

- ***reflexive*** – $R(a,a) = 1$, for all $a \in A$, $\Leftrightarrow$ $\Delta_A \leqslant R$
- ***symmetric*** – $R(a,b) = R(b,a)$, for all $a,b \in A$, $\Leftrightarrow$ $R^{-1} \leqslant R$
- ***transitive*** – $R(a,b) \otimes R(b,c) \leqslant R(a,c)$, for all $a,b,c \in A$, $\Leftrightarrow$ $R \circ R \leqslant R$

fuzzy quasi-order – *reflexive* and *transitive* (***fuzzy preorder***, in some sources)

fuzzy equivalence – *reflexive*, *symmetric* and *transitive*

Q – fuzzy quasi-order, $Q \wedge Q^{-1}$ – ***natural fuzzy equivalence*** of Q

$\mathcal{Q}(A)$ – complete ***lattice of fuzzy quasi-orders*** on A

$\mathcal{E}(A)$ – complete ***lattice of fuzzy equivalences*** on A

Fuzzy quasi-orders, fuzzy equivalences (cont.)

Closures of fuzzy relations

$R \in L^{A\times A}$ – fuzzy relation

$R^t = \bigvee_{n\in\mathbb{N}} R^n$ – *transitive closure* of R (the least transitive fuzzy relation containing R)

$|A| = n$ – $R^t = \bigvee_{k=1}^{n} R^k$, R is reflexive – $R^t = R^{n-1}$

$R^r = R \vee \Delta_A$ – *reflexive closure* of R

$R^s = R \vee R^{-1}$ – *symmetric closure* of R

$R^q = (R^r)^t = (R^t)^r$ – *fuzzy quasi-order closure* of R

(*reflexive-transitive closure*, $Q(A)$-*closure*)

$R^e = ((R^r)^s)^t = ((R^s)^r)^t = ((R^s)^t)^r$ – *fuzzy equivalence closure* of R

(*symmetric-reflexive-transitive closure*, $Q(A)$-*closure*)

Transitive closure – efficient algorithms

- Naessens, De Meyer, De Baets, IEEE Trans. Fuzzy Systems 10 (2002) 541–551.
  t-norm based structures – $O(n^3)$
- De Meyer, Naessens, De Baets, EJOR 155 (2004) 226–238.
  Gödel structure – $O(n^2)$

Fuzzy quasi-orders, fuzzy equivalences (cont.)

Aftersets, foresets, equivalence classes

$R \in L^{A\times A}$ – fuzzy relation

R-afterset of $a \in A$ is $aR \in L^A$ defined by: $(aR)(b) = R(a,b)$, for any $b \in A$,

R-foreset of a is $Ra \in L^A$ defined by: $(Ra)(b) = R(b,a)$, for any $b \in A$.

R is symmetric iff $aR = Ra$, for each $a \in A$;

if E is a fuzzy equivalence, we write E_a instead of $aE = Ea$;

E_a – ***equivalence class*** of a w.r.t. E

if $|A|$ is finite – *aftersets* $\equiv$ *row vectors*, *foresets* $\equiv$ *column vectors*

Fuzzy relations (cont.)

Residuals of fuzzy sets by fuzzy relations

$R \in L^{A\times A}$ – fuzzy relation, $f \in L^A$ – fuzzy subset

left residual $f/R \in L^A$ of f by R:

$$(f/R)(a) = \bigwedge_{b\in A} R(a,b) \to f(b) = \bigwedge_{b\in A} (aR)(b) \to f(b), \qquad (a \in A)$$

right residual $R\backslash f \in L^A$ of f by R:

$$(R\backslash f)(a) = \bigwedge_{b\in A} R(b,a) \to f(b) = \bigwedge_{b\in A} (Ra)(b) \to f(b), \qquad (a \in A)$$

concepts conjugated with compositions:

$$f \circ R \leqslant g \;\Leftrightarrow\; f \leqslant g/R, \qquad R \circ f \leqslant g \;\Leftrightarrow\; f \leqslant R\backslash g$$

if R is symmetric, then $f/R = R\backslash f$, for each $f \in L^A$

Fuzzy quasi-orders, fuzzy equivalences (cont.)

Theorem (Bodenhofer (2003), Bodenhofer, De Cock, Kerre (2003))

Let R be an arbitrary fuzzy relation on a set A. Then

(a) *Functions*

$$f \mapsto f \circ R, \qquad f \mapsto f/R, \qquad f \mapsto R \circ f, \qquad f \mapsto R\backslash f \tag{1}$$

are ***isotone*** *functions of the lattice $\mathcal{F}(A)$ into itself.*

(b) *If R is* ***reflexive****, then functions $f \mapsto f \circ R$ and $f \mapsto R \circ f$ are* ***extensive****, and functions $f \mapsto f/R$ and $f \mapsto R\backslash f$ are* ***intensive****.*

(c) *If R is a* ***fuzzy quasi-order****, then all functions in (1) are* ***idempotent****.*

In other words, If R is a ***fuzzy quasi-order****, then*

$f \mapsto f \circ R$ and $f \mapsto R \circ f$ are ***closure operators*** *on $\mathcal{F}(A)$;*

$f \mapsto f/R$ and $f \mapsto R\backslash f$ are ***opening operators*** *on $\mathcal{F}(A)$.*

Systems with left compositions and residuals

Notation

Σ_i $(i \in I)$ – system of fuzzy relation inequalities or equations

$\langle \Sigma_i \mid i \in I \rangle$ – the set of all solutions to this system

if $I = \{1, \ldots, n\}$, we write $\langle \Sigma_1, \ldots, \Sigma_n \rangle$

$\langle \Sigma \rangle$ – the set of all solutions to a single fuzzy relation inequality or equation Σ

Theorem

Let $\{R_i\}_{i \in I}$ be a family of fuzzy relations on a set A, let $R = \bigvee_{i \in I} R_i$ and $Q = R^q$, and let f denote an unknown fuzzy subset of A. Then

$$\begin{aligned}\langle f \circ R_i \leqslant f \mid i \in I \rangle &= \langle f \circ R \leqslant f \rangle = \langle f \circ Q \leqslant f \rangle = \langle f \circ Q = f \rangle \\ &= \langle f \leqslant f/R_i \mid i \in I \rangle = \langle f \leqslant f/R \rangle = \langle f \leqslant f/Q \rangle = \langle f = f/Q \rangle\end{aligned}$$

*Moreover, these sets consist of all fuzzy subsets of A that can be represented as **linear combinations of Q-aftersets**.*

Systems with right compositions and residuals, double systems

Theorem

Let $\{R_i\}_{i\in I}$ be a family of fuzzy relations on a set A, let $R = \bigvee_{i\in I} R_i$ and $Q = R^q$, and let f denote an unknown fuzzy subset of A. Then

$$\begin{aligned}\langle R_i \circ f \leqslant f \mid i \in I\rangle &= \langle R \circ f \leqslant f\rangle = \langle Q \circ f \leqslant f\rangle = \langle Q \circ f = f\rangle \\ &= \langle f \leqslant R_i\backslash f \mid i \in I\rangle = \langle f \leqslant R\backslash f\rangle = \langle f \leqslant Q\backslash f\rangle = \langle f = Q\backslash f\rangle\end{aligned}$$

Moreover, these sets consist of all fuzzy subsets of A that can be represented as ***linear combinations of Q-foresets****.*

Theorem

Let $\{R_i\}_{i\in I}$ be a family of fuzzy relations on a set A, let $R = \bigvee_{i\in I} R_i$ and $E = R^e$, and let f denote an unknown fuzzy subset of A. Then

$$\begin{aligned}\langle f \circ R_i \leqslant f, R_i \circ f \leqslant f \mid i \in I\rangle &= \langle f \circ R \leqslant f, R \circ f \leqslant f\rangle = \langle f \circ E \leqslant f\rangle \\ &= \langle f \circ E = f\rangle = \langle f \leqslant f/R_i, f \leqslant R_i\backslash f \mid i \in I\rangle \\ &= \langle f \leqslant f/R, f \leqslant R\backslash f\rangle = \langle f \leqslant f/E\rangle = \langle f = f/E\rangle\end{aligned}$$

Moreover, these sets consist of all fuzzy subsets of A that can be represented as ***linear combinations of E-classes****.*

Closedness w.r.t. a fuzzy relation

R-closed fuzzy subsets

$R \in L^{A\times A}$, $f \in L^A$

f is *left R-closed*: $f \circ R \leqslant f$ (iff $f(a) \otimes R(a,b) \leqslant f(b)$, for all $a, b \in A$)

other names: *R-closed*, *R-congruent*

Bodenhofer (2003), Bodenhofer, De Cock, Kerre (2003)

f is *right R-closed*: $R \circ f \leqslant f$ (iff $R(a,b) \otimes f(b) \leqslant f(a)$, for all $a, b \in A$)

double R-closed = left R-closed & right R-closed

R is symmetric: *f is left R-closed iff f is right R-closed* – we say just f is *R-closed*

E is a fuzzy equivalence: E-closed fuzzy subsets = fuzzy subsets *extensional w.r.t E*

Klawonn (2000)

Conjugation of systems

Theorem

Let $\{R_i\}_{i\in I}$ be a family of fuzzy relations on a set A, and let f be an unknown fuzzy subset of A. Then:

(i) *if* $g \in \langle f \circ R_i \leqslant f \mid i \in I \rangle$, *then* $g^{\neg} \in \langle R_i \circ f \leqslant f \mid i \in I \rangle$;

(ii) *if* $g \in \langle R_i \circ f \leqslant f \mid i \in I \rangle$, *then* $g^{\neg} \in \langle f \circ R_i \leqslant f \mid i \in I \rangle$;

(iii) *if* $g \in \langle f \circ R_i \leqslant f, R_i \circ f \leqslant f \mid i \in I \rangle$, *then* $g^{\neg} \in \langle f \circ R_i \leqslant f,\ R_i \circ f \leqslant f \mid i \in I \rangle$.

In addition, if the underlying complete residuated lattice $\mathcal{L}$ has the ***double negation property****, then the opposite implications also hold.*

Closure-opening systems

Theorem

Let $\{R_i\}_{i\in I}$ be a family of fuzzy relations on a set A, let $R = \bigvee_{i\in I} R_i$, $Q = R^q$ and $E = R^e$, and let f denote an unknown fuzzy subset of A. Then:

(i) $\langle f \circ R_i \leqslant f \mid i \in I\rangle$, $\langle R_i \circ f \leqslant f \mid i \in I\rangle$ *and* $\langle f \circ R_i \leqslant f,\ R_i \circ f \leqslant f \mid i \in I\rangle$ *are both* ***closure*** *and* ***opening systems*** *in* $\mathcal{F}(A)$.

(ii) *The principal part of* $\langle f \circ R_i \leqslant f \mid i \in I\rangle$ *consists of all* Q*-aftersets.*

(iii) *The principal part of* $\langle R_i \circ f \leqslant f \mid i \in I\rangle$ *consists of all* Q*-foresets.*

(iv) *The principal part of* $\langle f \circ R_i \leqslant f, R_i \circ f \leqslant f \mid i \in I\rangle$ *consists of all* E*-classes.*

Closure and opening operators

Closure and opening operators

$g \mapsto g^{lc}, g \mapsto g^{rc}, g \mapsto g^{dc}$ – *closure operators* on $\mathcal{F}(A)$ corresponding to closure systems

$$\langle f \circ R_i \leqslant f \mid i \in I\rangle, \quad \langle R_i \circ f \leqslant f \mid i \in I\rangle \quad \text{and} \quad \langle f \circ R_i \leqslant f,\ R_i \circ f \leqslant f \mid i \in I\rangle, \qquad (2)$$

$g \mapsto g^{lo}, g \mapsto g^{ro}, g \mapsto g^{do}$ – the corresponding *opening operators*

g^{lc}, g^{rc}, g^{dc} – the *least solutions* to systems in (2) containing g

g^{lo}, g^{ro}, g^{do} – the *greatest solutions* to systems in (2) contained in g

Theorem (Main Theorem)

Let $\{R_i\}_{i\in I}$ be a family of fuzzy relations on a set A, let $R = \bigvee_{i\in I} R_i$, $Q = R^q$ and $E = R^e$, and let g be an arbitrary fuzzy subset of A. Then:

$$g^{lc} = g \circ Q, \quad g^{rc} = Q \circ g, \quad g^{dc} = g \circ E, \quad g^{lo} = g/Q, \quad g^{ro} = Q\backslash g, \quad g^{do} = g/E.$$

Efficient algorithms

Efficient algorithms (transitive closure)

- compute $R = \bigvee_{i\in I} R_i$ and R^r (or $(R^r)^s$);
- compute the transitive closure of R^r (or $(R^r)^s$);
- compute the composition of a given fuzzy set g and $(R^r)^t$ (or $((R^r)^s)^t$).

computationally the most demanding part – computation of the ***transitive closure***

work independently of the properties of the underlying structure of truth values

Another method (Ignjatović, Ćirić, De Baets)

general method for solving inequalities defined by ***residuated*** and ***residual functions***

$f \circ R_i \leqslant f \ (i \in I)$ – defined by residuated functions $f \mapsto f \circ R_i$ and $f \mapsto f$, and residual functions $f \mapsto f/R_i$

$$\langle f \circ R_i \leqslant f \,|\, i \in I\rangle = \Big\langle \bigvee_{i\in I} (f \circ R_i) \leqslant f \Big\rangle = \langle f \leqslant f/R_i \,|\, i \in I\rangle = \Big\langle f \leqslant \bigwedge_{i\in I} (f/R_i) \Big\rangle$$

computation of the ***greatest post-fixed point*** or the ***least pre-fixed point*** of an isotone function on the lattice of fuzzy sets

iterative procedure (the number of iterations may be infinite, for some structures)

Comparison with the systems of inequalities

Theorem

Let $\{R_i\}_{i\in I}$ be a family of fuzzy relations on a set A, let $R = \bigvee_{i\in I} R_i$, and let f denote an unknown fuzzy subset of A. Then:

$$\langle f \circ R_i = f \mid i \in I\rangle \subseteq \langle f \circ R = f\rangle, \qquad \langle R_i \circ f = f \mid i \in I\rangle \subseteq \langle R \circ f = f\rangle, \tag{3}$$

and the inclusions may be proper.

Theorem

Let R be a fuzzy relation on a set A, and let f denote an unknown fuzzy subset of A. Then:

$$\langle f \circ R = f\rangle = \langle f \circ R^t = f\rangle, \qquad \langle R \circ f = f\rangle = \langle R^t \circ f = f\rangle. \tag{4}$$

Remark

$\langle f \circ R_i = f \mid i \in I\rangle$ and $\langle R_i \circ f = f \mid i \in I\rangle$ form *opening systems*, but *not necessarily closure systems*

Computation of the greatest solutions

Algorithm 1 (Ignjatović, Ćirić, De Baets)

general method for solving equations defined by ***residuated*** and ***residual functions***

$f \circ R_i = f$ $(i \in I)$ – defined by residuated functions $f \mapsto f \circ R_i$ and $f \mapsto f$

$$\langle f \circ R_i = f \mid i \in I \rangle = \Big\langle f \leqslant \bigwedge_{i \in I} (f \circ R_i \wedge f/R_i) \Big\rangle$$

computation of the ***greatest post-fixed point*** of an isotone function on $\mathcal{F}(A)$

iterative procedure (the number of iterations may be infinite, for some structures)

Algorithm 2 (Jiménez, Montes, Šešelja, Tepavčević, 2011)

compute the greatest solution to $f \circ R = f, f \leqslant g$ (R and g are given)

- compute the greatest solution h to $f \circ R \leqslant f, f \leqslant g$ $(h = g/R^q)$
- build a sequence $\{h_n\}_{n \in \mathbb{N}}$ of fuzzy sets by: $h_1 = h$, $h_{n+1} = h_n \circ R = h \circ R^n$
- if $h_n = h_{n+1}$, for some $n \in \mathbb{N}$, then h_n is the solution we were looking for

under what conditions the sequence $\{h_n\}_{n \in \mathbb{N}}$ is finite?