FUZZY RELATION EQUATIONS AND FUZZY AUTOMATA

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Fuzzy relation equations and inequalities

Employed in the study of fuzzy automata:

- in the study of simulation, bisimulation and equivalence of fuzzy automata,
- in the reduction of the number of states of fuzzy automata,
- in the study of subsystems of fuzzy transition systems...

These applications have led to the introduction of certain new types of fuzzy relation equations and inequalities, which have been investigated from the general point of view.

Fuzzy relations

$\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ – complete residuated lattice

$\varphi : A \times B \to L$ – fuzzy relation between sets A and B

$\mathcal{R}(A, B)$ – lattice of fuzzy relations between A and B

$\varphi : A \times A \to L$ – fuzzy relation on a set A

$\mathcal{R}(A)$ – lattice of fuzzy relations on A

$\varphi \circ \psi$ – composition of fuzzy relations $\varphi \in \mathcal{R}(A, B)$ and $\psi \in \mathcal{R}(B, C)$

$$(\varphi \circ \psi)(a, c) = \bigvee_{b \in B} \varphi(a, b) \otimes \psi(b, c) \qquad (a \in A,\ c \in C)$$

$\varphi^{-1} \in \mathcal{R}(B, A)$ – inverse of a fuzzy relation $\varphi \in \mathcal{R}(A, B)$

$$\varphi^{-1}(b, a) \stackrel{\text{def}}{=} \varphi(a, b) \qquad (a \in A,\ b \in B)$$

Basic concepts

$\mathcal{L}$ – complete residuated lattice, X – alphabet

Fuzzy automaton over $\mathcal{L}$ and X – a quadruple $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$

- A– (non-empty) set of states
- $\delta^A : A \times X \times A \to L$ fuzzy transition function
- $\sigma^A : A \to L$ fuzzy set of initial states
- $\tau^A : A \to L$ fuzzy set terminal states

X^* – the free monoid over the alphabet X
δ^A can be uniquely extended up to a mapping $\delta^A_* : A \times X^* \times A \to L$

fuzzy transition relation on A determined by $u \in X^*$

$$\delta^A_u(a, b) = \delta^A_*(a, u, b), \quad \text{for all } a, b \in A \tag{1}$$

$$\delta^A_{uv} = \delta^A_u \circ \delta^A_v, \quad \text{for all } u, v \in X^* \tag{2}$$

Bisimulations

Origin of bisimulation

concurrency theory – R. Milner, 1980, D. Park, 1981

modal logic (Kripke systems) – J. van Benthem, 1976

set theory – M. Forti, F. Honsell, 1983

Role of bisimulation – equivalence between two transition systems, reduction of the number of states

Recent research

M. Ćirić, J. Ignjatović, N. Damljanović, M. Bašić (FSS, 2012)

Bisimulations between fuzzy automata

M. Ćirić, J. Ignjatović, I. Jančić, N. Damljanović (FSS, to appear)

Computation of the greatest simulations and bisimulations

M. Ćirić, J. Ignjatović, M. Bašić, I. Jančić
(Comput. Math. Appl., to appear)

Bisimulations between nondeterministic automata

$\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$, $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ – fuzzy automata

$\varphi \in \mathcal{R}(A, B)$ – non-empty fuzzy relation

φ is a forward simulation if

$$\sigma^A \leqslant \sigma^B \circ \varphi^{-1}, \tag{fs-1}$$

$$\varphi^{-1} \circ \delta^A_x \leqslant \delta^B_x \circ \varphi^{-1}, \quad \text{for every } x \in X, \tag{fs-2}$$

$$\varphi^{-1} \circ \tau^A \leqslant \tau^B, \tag{fs-3}$$

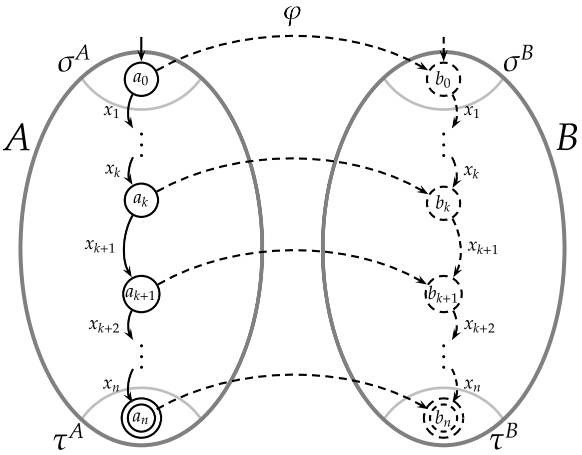
φ is a backward simulation if

$$\tau^A \leqslant \varphi \circ \tau^B, \tag{bs-1}$$

$$\delta^A_x \circ \varphi \leqslant \varphi \circ \delta^B_x, \quad \text{for every } x \in X, \tag{bs-2}$$

$$\sigma^A \circ \varphi \leqslant \sigma^B. \tag{bs-3}$$

Simulations between fuzzy automata



Homotypic bisimulations

φ is a forward bisimulation if both φ and φ^{-1} are forward simulations

$$\sigma^A \leqslant \sigma^B \circ \varphi^{-1}, \quad \sigma^B \leqslant \sigma^A \circ \varphi, \tag{fb-1}$$

$$\varphi^{-1} \circ \delta_x^A \leqslant \delta_x^B \circ \varphi^{-1}, \quad \varphi \circ \delta_x^B \leqslant \delta_x^A \circ \varphi, \quad \text{for every } x \in X, \tag{fb-2}$$

$$\varphi^{-1} \circ \tau^A \leqslant \tau^B, \quad \varphi \circ \tau^B \leqslant \tau^A, \tag{fb-3}$$

φ is a backward bisimulation, if both φ and φ^{-1} are backward simulations

$$\tau^A \leqslant \varphi \circ \tau^B, \quad \tau^B \leqslant \varphi^{-1} \circ \tau^A, \tag{bb-1}$$

$$\delta_x^A \circ \varphi \leqslant \varphi \circ \delta_x^B, \quad \delta_x^B \circ \varphi^{-1} \leqslant \varphi^{-1} \circ \delta_x^A, \quad \text{for every } x \in X, \tag{bb-2}$$

$$\sigma^A \circ \varphi \leqslant \sigma^B, \quad \sigma^B \circ \varphi^{-1} \leqslant \sigma^A. \tag{bb-3}$$

Heterotypic bisimulations

φ is a forward-backward bisimulation if

φ is a forward simulation and φ^{-1} is a backward simulation

$$\sigma^A \leqslant \sigma^B \circ \varphi^{-1}, \quad \tau^B \leqslant \varphi^{-1} \circ \tau^A, \qquad \text{(fbb-1)}$$

$$\varphi^{-1} \circ \delta_x^A = \delta_x^B \circ \varphi^{-1}, \qquad \text{for every } x \in X, \qquad \text{(fbb-2)}$$

$$\sigma^B \circ \varphi^{-1} \leqslant \sigma^A, \quad \varphi^{-1} \circ \tau^A \leqslant \tau^B, \qquad \text{(fbb-3)}$$

φ is a backward-forward bisimulation if

φ is a backward simulation and φ^{-1} is a forward simulation

$$\sigma^B \leqslant \sigma^A \circ \varphi, \quad \tau^A \leqslant \varphi \circ \tau^B, \qquad \text{(bfb-1)}$$

$$\delta_x^A \circ \varphi = \varphi \circ \delta_x^B, \qquad \text{for every } x \in X, \qquad \text{(bfb-2)}$$

$$\sigma^A \circ \varphi \leqslant \sigma^B, \quad \varphi \circ \tau^B \leqslant \tau^A. \qquad \text{(bfb-3)}$$

- Simulations $\Rightarrow$ languague inclusion
- Bisimulations $\Rightarrow$ languague equivalence
- Simulations and bisimulations $\Rightarrow$ compatibility with transitions, initial and terminal states

Heterotypic bisimulations

φ is a forward-backward bisimulation if

φ is a forward simulation and φ^{-1} is a backward simulation

$$\sigma^A \leqslant \sigma^B \circ \varphi^{-1}, \quad \tau^B \leqslant \varphi^{-1} \circ \tau^A, \tag{fbb-1}$$

$$\varphi^{-1} \circ \delta_x^A = \delta_x^B \circ \varphi^{-1}, \qquad \text{for every } x \in X, \tag{fbb-2}$$

$$\sigma^B \circ \varphi^{-1} \leqslant \sigma^A, \quad \varphi^{-1} \circ \tau^A \leqslant \tau^B, \tag{fbb-3}$$

φ is a backward-forward bisimulation if

φ is a backward simulation and φ^{-1} is a forward simulation

$$\sigma^B \leqslant \sigma^A \circ \varphi, \quad \tau^A \leqslant \varphi \circ \tau^B, \tag{bfb-1}$$

$$\delta_x^A \circ \varphi = \varphi \circ \delta_x^B, \qquad \text{for every } x \in X, \tag{bfb-2}$$

$$\sigma^A \circ \varphi \leqslant \sigma^B, \quad \varphi \circ \tau^B \leqslant \tau^A. \tag{bfb-3}$$

- Simulations $\Rightarrow$ languague inclusion
- Bisimulations $\Rightarrow$ languague equivalence
- Simulations and bisimulations $\Rightarrow$ compatibility with transitions, initial and terminal states

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ and $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ be fuzzy automata. We define fuzzy relations $\psi^w \in \mathcal{R}(A, B)$, for $w \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}\}$, in the following way:

$$
\begin{aligned}
\psi^{\mathrm{fs}} &= \tau^A \to \tau^B, \\
\psi^{\mathrm{bs}} &= \sigma^A \to \sigma^B, \\
\psi^{\mathrm{fb}} &= (\tau^A \to \tau^B) \wedge (\tau^A \leftarrow \tau^B) = \tau^A \leftrightarrow \tau^B, \\
\psi^{\mathrm{bb}} &= (\sigma^A \to \sigma^B) \wedge (\sigma^A \leftarrow \sigma^B) = \sigma^A \leftrightarrow \sigma^B, \\
\psi^{\mathrm{fbb}} &= (\tau^A \to \tau^B) \wedge (\sigma^A \leftarrow \sigma^B), \\
\psi^{\mathrm{bfb}} &= (\sigma^A \to \sigma^B) \wedge (\tau^A \leftarrow \tau^B).
\end{aligned}
$$

Moreover, we define functions $\phi^{w} : \mathcal{R}(A,B) \to \mathcal{R}(A,B)$, for $w \in \{fs, bs, fb, bb, fbb, bfb\}$, as follows:

$$\phi^{fs}(\alpha) = \bigwedge_{x\in X} [(\delta_x^B \circ \alpha^{-1})\backslash \delta_x^A]^{-1},$$

$$\phi^{bs}(\alpha) = \bigwedge_{x\in X} (\alpha \circ \delta_x^B)/\delta_x^A,$$

$$\phi^{fb}(\alpha) = \bigwedge_{x\in X} [(\delta_x^B \circ \alpha^{-1})\backslash \delta_x^A]^{-1} \wedge [(\delta_x^A \circ \alpha)\backslash \delta_x^B] = \phi^{fs}(\alpha) \wedge [\phi^{fs}(\alpha^{-1})]^{-1},$$

$$\phi^{bb}(\alpha) = \bigwedge_{x\in X} [(\alpha \circ \delta_x^B)/\delta_x^A] \wedge [(\alpha^{-1} \circ \delta_x^A)/\delta_x^B]^{-1} = \phi^{bs}(\alpha) \wedge [\phi^{bs}(\alpha^{-1})]^{-1},$$

$$\phi^{fbb}(\alpha) = \bigwedge_{x\in X} [(\delta_x^B \circ \alpha^{-1})\backslash \delta_x^A]^{-1} \wedge [(\alpha^{-1} \circ \delta_x^A)/\delta_x^B]^{-1} = \phi^{fs}(\alpha) \wedge [\phi^{bs}(\alpha^{-1})]^{-1},$$

$$\phi^{bfb}(\alpha) = \bigwedge_{x\in X} [(\alpha \circ \delta_x^B)/\delta_x^A] \wedge [(\delta_x^A \circ \alpha)\backslash \delta_x^B] = \phi^{bs}(\alpha) \wedge [\phi^{fs}(\alpha^{-1})]^{-1},$$

for any $\alpha \in \mathcal{R}(A,B)$. Notice that in the expression "$\phi^{w}(\alpha^{-1})$" ($w \in \{fs, bs\}$) we denote by ϕ^{w} a function from $\mathcal{R}(B,A)$ into itself.

Theorem

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ *and* $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ *be fuzzy automata and* $w \in \{fs, bs, fb, bb, fbb, bfb\}$*. A fuzzy relation* $\varphi \in \mathcal{R}(A, B)$ *satisfies conditions* $(w\text{-}2)$ *and* $(w\text{-}3)$ *if and only if it satisfies*

$$\varphi \leqslant \phi^w(\varphi), \qquad \varphi \leqslant \psi^w. \tag{3}$$

Boils down on computation of the greatest post-fixed point of the function ϕ^w

Theorem

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ *and* $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ *be fuzzy automata, let* $w \in \{fs, bs, fb, bb, fbb, bfb\}$*, and let a sequence* $\{\varphi_k\}_{k\in\mathbb{N}}$ *of fuzzy relations from* $\mathcal{R}(A, B)$ *be defined by*

$$\varphi_1 = \psi^w, \qquad \varphi_{k+1} = \varphi_k \wedge \phi^w(\varphi_k), \quad \text{for each } k \in \mathbb{N}. \tag{4}$$

If $\langle \mathrm{Im}(\psi^w) \cup \bigcup_{x\in X}(\mathrm{Im}(\delta^A_x) \cup \mathrm{Im}(\delta^B_x))\rangle$ *is a finite subalgebra of* $\mathcal{L}$*, then the following is true:*

(a) *the sequence* $\{\varphi_k\}_{k\in\mathbb{N}}$ *is finite and descending, and there is the least natural number* k *such that* $\varphi_k = \varphi_{k+1}$*;*

(b) φ_k *is the greatest fuzzy relation in* $\mathcal{R}(A, B)$ *which satisfies* $(w\text{-}2)$ *and* $(w\text{-}3)$*;*

(c) *if* φ_k *satisfies* $(w\text{-}1)$*, then it is the greatest fuzzy relation in* $\mathcal{R}(A, B)$ *which satisfies* $(w\text{-}1)$, $(w\text{-}2)$ *and* $(w\text{-}3)$*;*

(d) *if* φ_k *does not satisfy* $(w\text{-}1)$*, then there is no any fuzzy relation in* $\mathcal{R}(A, B)$ *which satisfies* $(w\text{-}1)$, $(w\text{-}2)$ *and* $(w\text{-}3)$*.*

Theorem

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ *and* $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ *be fuzzy automata, let* $w \in \{fs, bs, fb, bb, fbb, bfb\}$*, let* $\{\varphi_k\}_{k\in\mathbb{N}}$ *be a sequence of fuzzy relations from* $\mathcal{R}(A, B)$ *defined by* (4)*, and let*

$$\varphi = \bigwedge_{k\in\mathbb{N}} \varphi_k. \tag{5}$$

If $\mathcal{L}$ *is a complete residuated lattice satisfying the following conditions:*

$$x \vee \Big(\bigwedge_{i\in I} y_i\Big) = \bigwedge_{i\in I}(x \vee y_i) \quad \text{and} \quad x \otimes \Big(\bigwedge_{i\in I} y_i\Big) = \bigwedge_{i\in I}(x \otimes y_i), \tag{6}$$

for all $x \in L$ *and* $\{y_i\}_{i\in I} \subseteq L$*. Then the following is true:*

(a) φ *is the greatest fuzzy relation in* $\mathcal{R}(A, B)$ *which satisfies* $(w\text{-}2)$ *and* $(w\text{-}3)$*;*

(b) *if* φ *satisfies* $(w\text{-}1)$*, then it is the greatest fuzzy relation in* $\mathcal{R}(A, B)$ *which satisfies* $(w\text{-}1)$*,* $(w\text{-}2)$ *and* $(w\text{-}3)$*;*

(c) *if* φ *does not satisfy* $(w\text{-}1)$*, then there is no any fuzzy relation in* $\mathcal{R}(A, B)$ *which satisfies* $(w\text{-}1)$*,* $(w\text{-}2)$ *and* $(w\text{-}3)$*.*

Bisimulation equivalence relations

E – fuzzy equivalence relation on $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$
$(\Delta_A \leqslant E,\ E^{-1} \leqslant E,\ \ E \circ E \leqslant E)$

- E is forward bisimulation on $\mathcal{A}$,
- $E \circ \delta_x^A \leqslant \delta_x^A \circ E$, for any $x \in X$, and $E \circ \tau^A \leqslant \tau^A$,
- $E \circ \delta_x^A \circ E = \delta_x^A \circ E$, for any $x \in X$, and $E \circ \tau^A = \tau^A$.

Any fuzzy automaton has the greatest forward and backward simulations, which are fuzzy quasi-orders, and the greatest forward and backward bisimulations, which are fuzzy equivalence relations.

State reduction

Replace a fuzzy automaton by an equivalent fuzzy automaton with fewer states

equivalent $\equiv$ recognize the same fuzzy language

M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković (LNCS 2007; JCSS 2010)

reduction by fuzzy equivalences – factor fuzzy automata

A. Stamenković, M. Ćirić, J. Ignjatović (IS, to appear)

reduction by fuzzy quasi-orders – afterset (foreset) fuzzy automata

Uniform forward bisimulations

$R \in \mathcal{R}(A, B)$ – fuzzy relation

F. Klawonn, 2000 – partial fuzzy function

$R \circ R^{-1} \circ R \leqslant R$

M. Ćirić, J. Ignjatović, S. Bogdanović, 2009 – uniform fuzzy relation

complete, surjective, partial fuzzy function

$\Rightarrow \quad R \circ R^{-1} \circ R = R \qquad$ (also $\quad R^{-1} \circ R \circ R^{-1} = R^{-1}$)

uniform fuzzy relation $\equiv$ fuzzy equivalence between two sets

Theorem

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ and $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ be fuzzy automata and let $\varphi \in \mathcal{R}(A, B)$ be a uniform fuzzy relation. Then φ is a forward bisimulation if and only if the following hold:

$$\sigma^A \circ \varphi \circ \varphi^{-1} = \sigma^B \circ \varphi^{-1}, \qquad \sigma^A \circ \varphi = \sigma^B \circ \varphi^{-1} \circ \varphi, \tag{7}$$

$$\delta_x^A \circ \varphi \circ \varphi^{-1} = \varphi \circ \delta_x^B \circ \varphi^{-1}, \qquad \varphi^{-1} \circ \delta_x^A \circ \varphi = \delta_x^B \circ \varphi^{-1} \circ \varphi, \qquad x \in X, \tag{8}$$

$$\tau^A = \varphi \circ \tau^B, \qquad \varphi^{-1} \circ \tau^A = \tau^B. \tag{9}$$

Uniform forward bisimulations

Fuzzy equivalence relations on $\mathcal{A}$ and $\mathcal{B}$ induced by $\varphi \in \mathcal{R}(A, B)$.

Kernel of φ : $$E_A^\varphi(a_1, a_2) = \bigwedge_{b \in B} \varphi(a_1, b) \leftrightarrow \varphi(a_2, b), \text{for all } a_1, a_2 \in A \quad (10)$$

Co-kernel of φ : $$E_B^\varphi(b_1, b_2) = \bigwedge_{a \in A} \varphi(a, b_1) \leftrightarrow \varphi(a, b_2), \text{for all } b_1, b_2 \in B \quad (11)$$

Theorem

Let fuzzy automata $\mathcal{A} = (A, \sigma^A, X, \delta^A, \tau^A)$ and $\mathcal{B} = (B, \sigma^B, X, \delta^B, \tau^B)$ be UFB-equivalent.
Then there exists a uniform forward bisimulation φ between $\mathcal{A}$ and $\mathcal{B}$ whose kernel E_A^φ is the greatest forward bisimulation on $\mathcal{A}$ and the co-kernel E_B^φ is the greatest forward bisimulation on $\mathcal{B}$.
Moreover, φ is the greatest forward bisimulation between $\mathcal{A}$ and $\mathcal{B}$.

Uniform forward bisimulations

Theorem

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ *and* $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ *be fuzzy automata, and let* E *and* F *be the greatest forward bisimulations on* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively.*
Then $\mathcal{A}$ *and* $\mathcal{B}$ *are UFB-equivalent if and only if there exists an isomorphism* $\phi : \mathcal{A}/E \to \mathcal{B}/F$ *such that*

$$\widetilde{E}(E_{a_1}, E_{a_2}) = \widetilde{F}(\phi(E_{a_1}), \phi(E_{a_2})), \tag{12}$$

for all $a_1, a_2 \in A$.

Weakly linear systems

$\{V_i\}_{i\in I} \subseteq \mathcal{R}(A)$, $\{W_i\}_{i\in I} \subseteq \mathcal{R}(B)$ – families of fuzzy relations

$Z \subseteq \mathcal{R}(A,B)$ given fuzzy relation

U – unknown fuzzy relation in $\mathcal{R}(A,B)$

(1)	$U^{-1}\circ V_i \leqslant W_i \circ U^{-1}$	$(i\in I)$	$U \leqslant Z$
(2)	$V_i \circ U \leqslant U \circ W_i$	$(i\in I)$	$U \leqslant Z$

combinations of (1) and (2) (for U and U^{-1})

(3)	$U^{-1}\circ V_i \leqslant W_i \circ U^{-1}$	$U\circ W_i \leqslant V_i \circ U$	$(i\in I)$	$U \leqslant Z$
(4)	$V_i \circ U \leqslant U \circ W_i$	$W_i \circ U^{-1} \leqslant U^{-1}\circ V_i$	$(i\in I)$	$U \leqslant Z$
(5)	$V_i \circ U = U \circ W_i$		$(i\in I)$	$U \leqslant Z$
(6)	$U^{-1}\circ V_i = W_i \circ U^{-1}$		$(i\in I)$	$U \leqslant Z$

Weakly linear systems

$A = B \;\Rightarrow$ homogeneous weakly linear sistems
J. Ignjatović, M. Ćirić, S. Bogdanović (FSS, 2010)

- The systems (1)–(6) have the greatest solutions.
- If Z is a fuzzy quasi-order, then the greatest solutions to (1)–(3) are fuzzy quasi-orders.
- If Z is a fuzzy equivalence, then the greatest solutions to (4)–(6) are fuzzy equivalences.

$A \neq B \;\Rightarrow$ heterogeneous weakly linear sistems
J. Ignjatović, M. Ćirić, N. Damljanović, I. Jančić (FSS, 2012)

- All systems (1)–(6) have the greatest solutions (they may be empty).
- If Z is a partial fuzzy function, then the greatest solutions to (3) and (4) are partial fuzzy functions.

Computation of the greatest solutions

$$\phi^{(1)}(R) \stackrel{\text{def}}{=} \bigwedge_{i\in I} [(W_i \circ R^{-1})\backslash V_i]^{-1}$$

$$\phi^{(2)}(R) \stackrel{\text{def}}{=} \bigwedge_{i\in I} (R \circ W_i)/V_i$$

$$\phi^{(3)}(R) \stackrel{\text{def}}{=} \bigwedge_{i\in I} [(W_i \circ R^{-1})\backslash V_i]^{-1} \wedge [(V_i \circ R)\backslash W_i] = \phi^{(1)}(R) \wedge \phi^{(1)}(R^{-1})$$

$$\phi^{(4)}(R) \stackrel{\text{def}}{=} \bigwedge_{i\in I} [(R \circ W_i)/V_i] \wedge [(R^{-1} \circ V_i)/W_i]^{-1} = \phi^{(2)}(R) \wedge \phi^{(2)}(R^{-1})$$

$$\phi^{(5)}(R) \stackrel{\text{def}}{=} \bigwedge_{i\in I} [(R \circ W_i)/V_i] \wedge [(V_i \circ R)\backslash W_i] = \phi^{(2)}(R) \wedge \phi^{(1)}(R^{-1})$$

$$\phi^{(6)}(R) \stackrel{\text{def}}{=} \bigwedge_{i\in I} [(W_i \circ R^{-1})\backslash V_i]^{-1} \wedge [(R^{-1} \circ V_i)/W_i]^{-1} = \phi^{(1)}(R) \wedge \phi^{(2)}(R^{-1})$$

$$(Z/V)(a,b) = \bigwedge_{a'\in A} V(a',a) \to Z(a',b) \qquad (a \in A,\ b \in B)$$

$$(Z\backslash W)(a,b) = \bigwedge_{b'\in B} W(b,b') \to Z(a,b') \qquad (a \in A,\ b \in B)$$

The greatest solutions to (3) and (4) – partial fuzzy functions $(R \circ R^{-1} \circ R \leqslant R)$

Question: When the greatest solution is a uniform fuzzy relation?

If there is a uniform solution, then the greatest solution is uniform

uniform solution – some kind of equivalence between fuzzy relational systems $(A, \{V_i\}_{i \in I})$ and $(B, \{W_i\}_{i \in I})$

H – uniform solution

H determines an isomorphism between factor fuzzy relational systems of $(A, \{V_i\}_{i \in I})$ and $(B, \{W_i\}_{i \in I})$ w.r.t. fuzzy equivalences $H \circ H^{-1}$ and $H^{-1} \circ H$