FUZZY RELATION EQUATIONS
AND FUZZY AUTOMATA

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Fuzzy relation equations and inequalities

Employed in the study of fuzzy automata:

• in the study of simulation, bisimulation and equivalence of fuzzy automata,
• in the reduction of the number of states of fuzzy automata,
• in the study of subsystems of fuzzy transition systems...

These applications have led to the introduction of certain new types of fuzzy relation equations and inequalities, which have been investigated from the general point of view.
Basic concepts

Fuzzy relations

\( \mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1) \) – complete residuated lattice

\( \varphi : A \times B \rightarrow L \) – fuzzy relation between sets \( A \) and \( B \)

\( \mathcal{R}(A, B) \) – lattice of fuzzy relations between \( A \) and \( B \)

\( \varphi : A \times A \rightarrow L \) – fuzzy relation on a set \( A \)

\( \mathcal{R}(A) \) – lattice of fuzzy relations on \( A \)

\( \varphi \circ \psi \) – composition of fuzzy relations \( \varphi \in \mathcal{R}(A, B) \) and \( \psi \in \mathcal{R}(B, C) \)

\[
(\varphi \circ \psi)(a, c) = \bigvee_{b \in B} \varphi(a, b) \otimes \psi(b, c) \quad (a \in A, \ c \in C)
\]

\( \varphi^{-1} \in \mathcal{R}(B, A) \) – inverse of a fuzzy relation \( \varphi \in \mathcal{R}(A, B) \)

\[
\varphi^{-1}(b, a) \overset{\text{def}}{=} \varphi(a, b) \quad (a \in A, \ b \in B)
\]
Basic concepts

\( \mathcal{L} \) – complete residuated lattice, \( X \) – alphabet

**Fuzzy automaton over** \( \mathcal{L} \) and \( X \) – a quadruple \( A = (A, \delta^A, \sigma^A, \tau^A) \)

- \( A \) – (non-empty) set of states
- \( \delta^A : A \times X \times A \rightarrow \mathcal{L} \) fuzzy transition function
- \( \sigma^A : A \rightarrow \mathcal{L} \) fuzzy set of initial states
- \( \tau^A : A \rightarrow \mathcal{L} \) fuzzy set terminal states

\( X^* \) – the free monoid over the alphabet \( X \)
\( \delta^A \) can be uniquely extended up to a mapping \( \delta^A_* : A \times X^* \times A \rightarrow \mathcal{L} \)

**Fuzzy transition relation** on \( A \) determined by \( u \in X^* \)

\[
\delta^A_u(a, b) = \delta^A_*(a, u, b), \quad \text{for all } a, b \in A
\]  

(1)

\[
\delta^A_{uv} = \delta^A_u \circ \delta^A_v, \quad \text{for all } u, v \in X^*
\]  

(2)
Bisimulations

Origin of bisimulation


modal logic (Kripke systems) – J. van Benthem, 1976

set theory – M. Forti, F. Honsell, 1983

Role of bisimulation – equivalence between two transition systems, reduction of the number of states

Recent research

M. Ćirić, J. Ignjatović, N. Damljanović, M. Bašić (FSS, 2012)  
Bisimulations between fuzzy automata

Computation of the greatest simulations and bisimulations

M. Ćirić, J. Ignjatović, M. Bašić, I. Jančić  
Bisimulations between nondeterministic automata
Simulations between fuzzy automata

\[ A = (A, \delta^A, \sigma^A, \tau^A), \ B = (B, \delta^B, \sigma^B, \tau^B) \] – fuzzy automata

\[ \varphi \in \mathcal{R}(A, B) \] – non-empty fuzzy relation

\( \varphi \) is a **forward simulation** if

\[
\sigma^A \leq \sigma^B \circ \varphi^{-1}, \tag{fs-1}
\]

\[
\varphi^{-1} \circ \delta^A_x \leq \delta^B_x \circ \varphi^{-1}, \quad \text{for every } x \in X, \tag{fs-2}
\]

\[
\varphi^{-1} \circ \tau^A \leq \tau^B, \tag{fs-3}
\]

\( \varphi \) is a **backward simulation** if

\[
\tau^A \leq \varphi \circ \tau^B, \tag{bs-1}
\]

\[
\delta^A_x \circ \varphi \leq \varphi \circ \delta^B_x, \quad \text{for every } x \in X, \tag{bs-2}
\]

\[
\sigma^A \circ \varphi \leq \sigma^B. \tag{bs-3}
\]
Simulations between fuzzy automata

A

B

\( \sigma^A \)

\( \sigma^B \)

\( \tau^A \)

\( \tau^B \)

\( \varphi \)

\( a_0 \)

\( a_k \)

\( a_{k+1} \)

\( a_n \)

\( b_0 \)

\( b_k \)

\( b_{k+1} \)

\( b_n \)

\( x_1 \)

\( x_k \)

\( x_{k+1} \)

\( x_{k+2} \)

\( x_n \)
Bisimulations between fuzzy automata

Homotypic bisimulations

φ is a **forward bisimulation** if both φ and φ⁻¹ are forward simulations

\[
\sigma^A \leq \sigma^B \circ \varphi^{-1}, \quad \sigma^B \leq \sigma^A \circ \varphi, \quad (fb-1)
\]

\[
\varphi^{-1} \circ \delta^A_x \leq \delta^B_x \circ \varphi^{-1}, \quad \varphi \circ \delta^B_x \leq \delta^A_x \circ \varphi, \quad \text{for every } x \in X, \quad (fb-2)
\]

\[
\varphi^{-1} \circ \tau^A \leq \tau^B, \quad \varphi \circ \tau^B \leq \tau^A, \quad (fb-3)
\]

φ is a **backward bisimulation**, if both φ and φ⁻¹ are backward simulations

\[
\tau^A \leq \varphi \circ \tau^B, \quad \tau^B \leq \varphi^{-1} \circ \tau^A, \quad (bb-1)
\]

\[
\delta^A_x \circ \varphi \leq \varphi \circ \delta^B_x, \quad \delta^B_x \circ \varphi^{-1} \leq \varphi^{-1} \circ \delta^A_x, \quad \text{for every } x \in X, \quad (bb-2)
\]

\[
\sigma^A \circ \varphi \leq \sigma^B, \quad \sigma^B \circ \varphi^{-1} \leq \sigma^A. \quad (bb-3)
\]
Heterotypic bisimulations

ϕ is a **forward-backward bisimulation** if

ϕ is a forward simulation and ϕ⁻¹ is a backward simulation

\[ \sigma^A \leq \sigma^B \circ \varphi^{-1}, \quad \tau^B \leq \varphi^{-1} \circ \tau^A, \]  
\[ \varphi^{-1} \circ \delta^A_x = \delta^B_x \circ \varphi^{-1}, \quad \text{for every } x \in X, \]  
\[ \sigma^B \circ \varphi^{-1} \leq \sigma^A, \quad \varphi^{-1} \circ \tau^A \leq \tau^B, \]  

ϕ is a **backward-forward bisimulation** if

ϕ is a backward simulation and ϕ⁻¹ is a forward simulation

\[ \sigma^B \leq \sigma^A \circ \varphi, \quad \tau^A \leq \varphi \circ \tau^B, \]  
\[ \delta^A_x \circ \varphi = \varphi \circ \delta^B_x, \quad \text{for every } x \in X, \]  
\[ \sigma^A \circ \varphi \leq \sigma^B, \quad \varphi \circ \tau^B \leq \tau^A. \]  

- Simulations ⇒ language inclusion
- Bisimulations ⇒ language equivalence
- Simulations and bisimulations ⇒ compatibility with transitions, initial and terminal states
Heterotypic bisimulations

\( \varphi \) is a forward-backward bisimulation if

\( \varphi \) is a forward simulation and \( \varphi^{-1} \) is a backward simulation

\[
\sigma^A \leq \sigma^B \circ \varphi^{-1}, \quad \tau^B \leq \varphi^{-1} \circ \tau^A, \tag{fbb-1}
\]

\[
\varphi^{-1} \circ \delta^A_x = \delta^B_x \circ \varphi^{-1}, \quad \text{for every } x \in X, \tag{fbb-2}
\]

\[
\sigma^B \circ \varphi^{-1} \leq \sigma^A, \quad \varphi^{-1} \circ \tau^A \leq \tau^B, \tag{fbb-3}
\]

\( \varphi \) is a backward-forward bisimulation if

\( \varphi \) is a backward simulation and \( \varphi^{-1} \) is a forward simulation

\[
\sigma^B \leq \sigma^A \circ \varphi, \quad \tau^A \leq \varphi \circ \tau^B, \tag{bfb-1}
\]

\[
\delta^A_x \circ \varphi = \varphi \circ \delta^B_x, \quad \text{for every } x \in X, \tag{bfb-2}
\]

\[
\sigma^A \circ \varphi \leq \sigma^B, \quad \varphi \circ \tau^B \leq \tau^A. \tag{bfb-3}
\]

- Simulations \( \Rightarrow \) language inclusion
- Bisimulations \( \Rightarrow \) language equivalence
- Simulations and bisimulations \( \Rightarrow \) compatibility with transitions, initial and terminal states
Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ and $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ be fuzzy automata. We define fuzzy relations $\psi^w \in \mathcal{R}(A, B)$, for $w \in \{fs, bs, fb, bb, fbb, bfb\}$, in the following way:

\[
\begin{align*}
\psi^{fs} &= \tau^A \rightarrow \tau^B, \\
\psi^{bs} &= \sigma^A \rightarrow \sigma^B, \\
\psi^{fb} &= (\tau^A \rightarrow \tau^B) \land (\tau^A \leftarrow \tau^B) = \tau^A \leftrightarrow \tau^B, \\
\psi^{bb} &= (\sigma^A \rightarrow \sigma^B) \land (\sigma^A \leftarrow \sigma^B) = \sigma^A \leftrightarrow \sigma^B, \\
\psi^{fbb} &= (\tau^A \rightarrow \tau^B) \land (\sigma^A \leftarrow \sigma^B), \\
\psi^{bfb} &= (\sigma^A \rightarrow \sigma^B) \land (\tau^A \leftarrow \tau^B).
\end{align*}
\]
Computation of the greatest simulations and bisimulations

Moreover, we define functions $\phi^w : \mathcal{R}(A, B) \to \mathcal{R}(A, B)$, for $w \in \{fs, bs, fb, bb, fbb, bfb\}$, as follows:

\[
\begin{align*}
\phi^{fs}(\alpha) &= \bigwedge_{x \in X} [(\delta^B_x \circ \alpha^{-1}) \setminus \delta^A_x]^{-1}, \\
\phi^{bs}(\alpha) &= \bigwedge_{x \in X} (\alpha \circ \delta^B_x)/\delta^A_x, \\
\phi^{fb}(\alpha) &= \bigwedge_{x \in X} [(\delta^B_x \circ \alpha^{-1}) \setminus \delta^A_x]^{-1} \land [(\delta^A_x \circ \alpha) \setminus \delta^B_x] = \phi^{fs}(\alpha) \land [\phi^{fs}(\alpha^{-1})]^{-1}, \\
\phi^{bb}(\alpha) &= \bigwedge_{x \in X} [(\alpha \circ \delta^B_x)/\delta^A_x] \land [(\alpha^{-1} \circ \delta^A_x)/\delta^B_x]^{-1} = \phi^{bs}(\alpha) \land [\phi^{bs}(\alpha^{-1})]^{-1}, \\
\phi^{fbb}(\alpha) &= \bigwedge_{x \in X} [(\delta^B_x \circ \alpha^{-1}) \setminus \delta^A_x]^{-1} \land [(\alpha^{-1} \circ \delta^A_x) \setminus \delta^B_x]^{-1} = \phi^{fs}(\alpha) \land [\phi^{bs}(\alpha^{-1})]^{-1}, \\
\phi^{bfb}(\alpha) &= \bigwedge_{x \in X} [(\alpha \circ \delta^B_x)/\delta^A_x] \land [(\delta^A_x \circ \alpha) \setminus \delta^B_x] = \phi^{bs}(\alpha) \land [\phi^{fs}(\alpha^{-1})]^{-1},
\end{align*}
\]

for any $\alpha \in \mathcal{R}(A, B)$. Notice that in the expression “$\phi^w(\alpha^{-1})$” ($w \in \{fs, bs\}$) we denote by $\phi^w$ a function from $\mathcal{R}(B, A)$ into itself.
Theorem

Let $A = (A, \delta^A, \sigma^A, \tau^A)$ and $B = (B, \delta^B, \sigma^B, \tau^B)$ be fuzzy automata and $\omega \in \{fs, bs, fb, bb, fbb, bfb\}$. A fuzzy relation $\varphi \in \mathcal{R}(A, B)$ satisfies conditions (\omega-2) and (\omega-3) if and only if it satisfies

$$\varphi \leq \phi^\omega(\varphi), \quad \varphi \leq \psi^\omega.$$  \hspace{1cm} (3)

Boils down on computation of the greatest post-fixed point of the function $\phi^\omega$
Computation of the greatest simulations and bisimulations

Theorem

Let \( A = (A, \delta^A, \sigma^A, \tau^A) \) and \( B = (B, \delta^B, \sigma^B, \tau^B) \) be fuzzy automata, let \( w \in \{fs, bs, fb, bb, fbb, bfb\} \), and let a sequence \( \{\varphi_k\}_{k \in \mathbb{N}} \) of fuzzy relations from \( R(A, B) \) be defined by

\[
\varphi_1 = \psi^w, \quad \varphi_{k+1} = \varphi_k \wedge \varphi^w(\varphi_k), \quad \text{for each } k \in \mathbb{N}. \tag{4}
\]

If \( \langle \text{Im}(\psi^w) \cup \bigcup_{x \in X} (\text{Im}(\delta^A_x) \cup \text{Im}(\delta^B_x)) \rangle \) is a finite subalgebra of \( \mathcal{L} \), then the following is true:

(a) the sequence \( \{\varphi_k\}_{k \in \mathbb{N}} \) is finite and descending, and there is the least natural number \( k \) such that \( \varphi_k = \varphi_{k+1} \);

(b) \( \varphi_k \) is the greatest fuzzy relation in \( R(A, B) \) which satisfies \((w-2)\) and \((w-3)\);

(c) if \( \varphi_k \) satisfies \((w-1)\), then it is the greatest fuzzy relation in \( R(A, B) \) which satisfies \((w-1)\), \((w-2)\) and \((w-3)\);

(d) if \( \varphi_k \) does not satisfy \((w-1)\), then there is no any fuzzy relation in \( R(A, B) \) which satisfies \((w-1)\), \((w-2)\) and \((w-3)\).
Theorem

Let $A = (A, \delta^A, \sigma^A, \tau^A)$ and $B = (B, \delta^B, \sigma^B, \tau^B)$ be fuzzy automata, let $w \in \{fs, bs, fb, bb, fbb, bfb\}$, let $\{\varphi_k\}_{k \in \mathbb{N}}$ be a sequence of fuzzy relations from $\mathcal{R}(A, B)$ defined by (4), and let

$$\varphi = \bigwedge_{k \in \mathbb{N}} \varphi_k. \quad (5)$$

If $\mathcal{L}$ is a complete residuated lattice satisfying the following conditions:

$$x \lor (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \lor y_i) \quad \text{and} \quad x \otimes (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \otimes y_i), \quad (6)$$

for all $x \in \mathcal{L}$ and $\{y_i\}_{i \in I} \subseteq \mathcal{L}$. Then the following is true:

(a) $\varphi$ is the greatest fuzzy relation in $\mathcal{R}(A, B)$ which satisfies $(w-2)$ and $(w-3)$;

(b) if $\varphi$ satisfies $(w-1)$, then it is the greatest fuzzy relation in $\mathcal{R}(A, B)$ which satisfies $(w-1)$, $(w-2)$ and $(w-3)$;

(c) if $\varphi$ does not satisfy $(w-1)$, then there is no any fuzzy relation in $\mathcal{R}(A, B)$ which satisfies $(w-1)$, $(w-2)$ and $(w-3)$.
Bisimulation equivalence relations

\( E \) – fuzzy equivalence relation on \( \mathcal{A} = (\mathcal{A}, \delta^\mathcal{A}, \sigma^\mathcal{A}, \tau^\mathcal{A}) \)

\( (\Delta^\mathcal{A} \leq E, E^{-1} \leq E, E \circ E \leq E) \)

- \( E \) is forward bisimulation on \( \mathcal{A} \),
- \( E \circ \delta^\mathcal{A}_x \leq \delta^\mathcal{A}_x \circ E \), for any \( x \in X \), and \( E \circ \tau^\mathcal{A} \leq \tau^\mathcal{A} \),
- \( E \circ \delta^\mathcal{A}_x \circ E = \delta^\mathcal{A}_x \circ E \), for any \( x \in X \), and \( E \circ \tau^\mathcal{A} = \tau^\mathcal{A} \).

Any fuzzy automaton has the greatest forward and backward simulations, which are fuzzy quasi-orders, and the greatest forward and backward bisimulations, which are fuzzy equivalence relations.

State reduction

Replace a fuzzy automaton by an equivalent fuzzy automaton with fewer states equivalent \( \equiv \) recognize the same fuzzy language

M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković (LNCS 2007; JCSS 2010)

reduction by fuzzy equivalences – factor fuzzy automata

A. Stamenković, M. Ćirić, J. Ignjatović (IS, to appear)

reduction by fuzzy quasi-orders – afterset (foreset) fuzzy automata
Uniform forward bisimulations

**R ∈ R(A, B)** – fuzzy relation

F. Klawonn, 2000 – partial fuzzy function

\[ R \circ R^{-1} \circ R \leq R \]

M. Ćirić, J. Ignjatović, S. Bogdanović, 2009 – uniform fuzzy relation

complete, surjective, partial fuzzy function

⇒ \[ R \circ R^{-1} \circ R = R \] (also \[ R^{-1} \circ R \circ R^{-1} = R^{-1} \])

uniform fuzzy relation ≡ fuzzy equivalence between two sets

**Theorem**

Let \( A = (A, \delta^A, \sigma^A, \tau^A) \) and \( B = (B, \delta^B, \sigma^B, \tau^B) \) be fuzzy automata and let \( \varphi \in R(A, B) \) be a uniform fuzzy relation. Then \( \varphi \) is a forward bisimulation if and only if the following hold:

\[ \sigma^A \circ \varphi \circ \varphi^{-1} = \sigma^B \circ \varphi^{-1}, \quad \sigma^A \circ \varphi = \sigma^B \circ \varphi^{-1} \circ \varphi, \]

\[ \delta^A_x \circ \varphi \circ \varphi^{-1} = \varphi \circ \delta^B_x \circ \varphi^{-1}, \quad \varphi^{-1} \circ \delta^A_x \circ \varphi = \delta^B_x \circ \varphi^{-1} \circ \varphi, \quad x \in X, \]

\[ \tau^A = \varphi \circ \tau^B, \quad \varphi^{-1} \circ \tau^A = \tau^B. \]
Uniform forward bisimulations

Fuzzy equivalence relations on \( A \) and \( B \) induced by \( \varphi \in \mathcal{R}(A, B) \).

**Kernel** of \( \varphi \):
\[
E^\varphi_A(a_1, a_2) = \bigwedge_{b \in B} \varphi(a_1, b) \leftrightarrow \varphi(a_2, b), \text{ for all } a_1, a_2 \in A
\] (10)

**Co-kernel** of \( \varphi \):
\[
E^\varphi_B(b_1, b_2) = \bigwedge_{a \in A} \varphi(a, b_1) \leftrightarrow \varphi(a, b_2), \text{ for all } b_1, b_2 \in B
\] (11)

**Theorem**

Let fuzzy automata \( A = (A, \sigma^A, X, \delta^A, \tau^A) \) and \( B = (B, \sigma^B, X, \delta^B, \tau^B) \) be UFB-equivalent. Then there exists a uniform forward bisimulation \( \varphi \) between \( A \) and \( B \) whose kernel \( E^\varphi_A \) is the greatest forward bisimulation on \( A \) and the co-kernel \( E^\varphi_B \) is the greatest forward bisimulation on \( B \). Moreover, \( \varphi \) is the greatest forward bisimulation between \( A \) and \( B \).
Theorem

Let $A = (A, \delta^A, \sigma^A, \tau^A)$ and $B = (B, \delta^B, \sigma^B, \tau^B)$ be fuzzy automata, and let $E$ and $F$ be the greatest forward bisimulations on $A$ and $B$, respectively. Then $A$ and $B$ are UFB-equivalent if and only if there exists an isomorphism $\phi : A/E \to B/F$ such that

$$\tilde{E}(E_{a_1}, E_{a_2}) = \tilde{F}(\phi(E_{a_1}), \phi(E_{a_2})),$$

for all $a_1, a_2 \in A$. 

(12)
Weakly linear systems

\{V_i\}_{i \in I} \subseteq \mathcal{R}(A), \quad \{W_i\}_{i \in I} \subseteq \mathcal{R}(B) – families of fuzzy relations

Z \subseteq \mathcal{R}(A, B) given fuzzy relation

U – unknown fuzzy relation in \mathcal{R}(A, B)

(1) \quad U^{-1} \circ V_i \leq W_i \circ U^{-1} \quad (i \in I) \quad U \leq Z

(2) \quad V_i \circ U \leq U \circ W_i \quad (i \in I) \quad U \leq Z

combinations of (1) and (2) (for U and U^{-1})

(3) \quad U^{-1} \circ V_i \leq W_i \circ U^{-1} \quad U \circ W_i \leq V_i \circ U \quad (i \in I) \quad U \leq Z

(4) \quad V_i \circ U \leq U \circ W_i \quad W_i \circ U^{-1} \leq U^{-1} \circ V_i \quad (i \in I) \quad U \leq Z

(5) \quad V_i \circ U = U \circ W_i \quad (i \in I) \quad U \leq Z

(6) \quad U^{-1} \circ V_i = W_i \circ U^{-1} \quad (i \in I) \quad U \leq Z
Weakly linear systems

\( A = B \Rightarrow \) homogeneous weakly linear systems
J. Ignjatović, M. Ćirić, S. Bogdanović (FSS, 2010)

- The systems (1)–(6) have the greatest solutions.
- If \( Z \) is a fuzzy quasi-order, then the greatest solutions to (1)–(3) are fuzzy quasi-orders.
- If \( Z \) is a fuzzy equivalence, then the greatest solutions to (4)–(6) are fuzzy equivalences.

\( A \neq B \Rightarrow \) heterogeneous weakly linear systems

- All systems (1)–(6) have the greatest solutions (they may be empty).
- If \( Z \) is a partial fuzzy function, then the greatest solutions to (3) and (4) are partial fuzzy functions.
Computation of the greatest solutions

\[ \phi^{(1)}(R) \overset{\text{def}}{=} \bigwedge_{i \in I} [(W_i \circ R^{-1}) \setminus V_i]^{-1} \]

\[ \phi^{(2)}(R) \overset{\text{def}}{=} \bigwedge_{i \in I} (R \circ W_i) / V_i \]

\[ \phi^{(3)}(R) \overset{\text{def}}{=} \bigwedge_{i \in I} [(W_i \circ R^{-1}) \setminus V_i]^{-1} \land [(V_i \circ R) \setminus W_i] = \phi^{(1)}(R) \land \phi^{(1)}(R^{-1}) \]

\[ \phi^{(4)}(R) \overset{\text{def}}{=} \bigwedge_{i \in I} [(R \circ W_i) / V_i] \land [(R^{-1} \circ V_i) / W_i]^{-1} = \phi^{(2)}(R) \land \phi^{(2)}(R^{-1}) \]

\[ \phi^{(5)}(R) \overset{\text{def}}{=} \bigwedge_{i \in I} [(R \circ W_i) / V_i] \land [(V_i \circ R) \setminus W_i] = \phi^{(2)}(R) \land \phi^{(1)}(R^{-1}) \]

\[ \phi^{(6)}(R) \overset{\text{def}}{=} \bigwedge_{i \in I} [(W_i \circ R^{-1}) \setminus V_i]^{-1} \land [(R^{-1} \circ V_i) / W_i]^{-1} = \phi^{(1)}(R) \land \phi^{(2)}(R^{-1}) \]

\[ (Z/V)(a, b) = \bigwedge_{a' \in A} V(a', a) \rightarrow Z(a', b) \quad (a \in A, \ b \in B) \]

\[ (Z\setminus W)(a, b) = \bigwedge_{b' \in B} W(b, b') \rightarrow Z(a, b') \quad (a \in A, \ b \in B) \]
Meaning of uniform solutions

The greatest solutions to (3) and (4) – partial fuzzy functions \( (R \circ R^{-1} \circ R \leq R) \)

Question: When the greatest solution is a uniform fuzzy relation?

*If there is a uniform solution, then the greatest solution is uniform*

uniform solution – some kind of equivalence between fuzzy relational systems \((A, \{V_i\}_{i \in I})\) and \((B, \{W_i\}_{i \in I})\)

\(H\) – uniform solution

\(H\) determines an isomorphism between factor fuzzy relational systems of \((A, \{V_i\}_{i \in I})\) and \((B, \{W_i\}_{i \in I})\) w.r.t. fuzzy equivalences \(H \circ H^{-1}\) and \(H^{-1} \circ H\)