Extensions of fuzzy sets in knowledge representation

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GIARA

Outline

Origin

1. Classical logic
   - Truth/Falsity. Boole's algebra. Non-contradiction principle and excluded middle principle.
2. Pierce's triadic logic.
3. Brouwer's **intuitionistic** logic (1907)
   - Constructibility:
     - Objects are mental intuitions.
     - Properties of objects are the properties proposed in their construction.
   - Intuitionistic propositional calculus. Heyting algebras(1930)
   - Neither double negation nor middle excluded principle are fulfilled.
   - 1984 new sets.
4. Multivalued logics. J. Łukasiewicz (1878-1956)
   - MV-algebras. The middle excluded principle is not valid.

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Fuzzy sets

L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353: A fuzzy set is a class with a continuum of membership grades.

Definition

A fuzzy set A *over a referential set* U *is an object:*

$$A = \{(u_i, \mu_A(u_i)) | u_i \in U\}$$

where $\mu_A : U \longrightarrow [0, 1]$.

$\mu_A(u_i)$ represents the degree of membership of the element $u_i \in U$ to the set A.

Fuzzy sets

$$FS(U) \equiv [0,1]^U \qquad \mu_A(u_i) \equiv A(u_i)$$

$$A \cup B(u_i) = max(A(u_i), B(u_i))$$
$$A \cap B(u_i) = min(A(u_i), B(u_i))$$

$(FS(U), \cup, \cap)$ is a complete lattice

$A \leq_{FS} B$ if and only if $A(u_i) \leq B(u_i)$ for every $u_i \in U$

- R.C. Willmott, Mean measures in fuzzy power-set theory, Report No. FRP-6, Department of Mathematics, University of Essex, Colchester, CO4 3SQ, England (1979)
- W. Bandler, L. Kohout, Fuzzy power sets, fuzzy implication operators, Fuzzy Sets Syst. 4 (1980) 1330

- J. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145-174

Definition

Let $(L, \vee, \wedge)$ be a complete lattice. A L-fuzzy set over the referential set U is a mapping

$$A : U \to L$$

$$A \cup B(u_i) = \vee(A(u_i), B(u_i))$$
$$A \cap B(u_i) = \wedge(A(u_i), B(u_i))$$

$(L\text{-}FS(U), \cup, \cap)$ is a complete lattice

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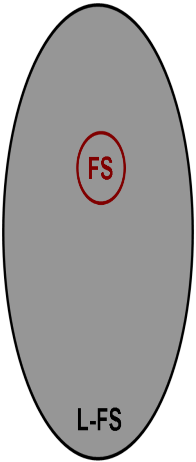

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Origin of the extensions

On December 11, 2008, L. A. Zadeh said:

Fuzzy logic is a precise reasoning, deduction and computation system in which the discourse and analysis objects are associated to information which is, or can be, imperfect.

Imperfect ionformation is defined as that information that in one or more senses is imprecise, uncertain, vague, incomplete, partially true or partially possible.

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Origin of the extensions

1. In fuzzy logic, everything is, or can be, a question of degree. **Such degrees may be fuzzy**

2. Fuzzy logic is not a replacement for bivalued logic or for probability based in bivalued logic. Fuzzy logic adds to bivalued logic and probability based in bivalued logic a wide spectrum of concepts and techniques for treating imperfect information.

3. Fuzzy logic intends to handle reasoning, deduction and computing problems with imperfect information which are unreachable for traditional methods based in bivalued logic and bivalued logic-based probability.

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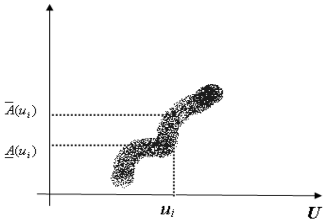
Origin of the extensions

4 In fuzzy logic, the writing tool is a painting spray with a **perfectly known and adjustable spray pattern**. In bivalued logic the writing tool is a pen.



5 The importance of fuzzy logic comes from the fact that, in most cases, in real world, imperfect information is the rule rather than the exception.

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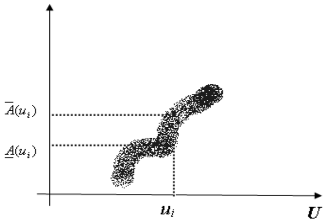

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Origin of the extensions

1. Fuzzy sets arise to handle dicourse objects associated to imperfect information.

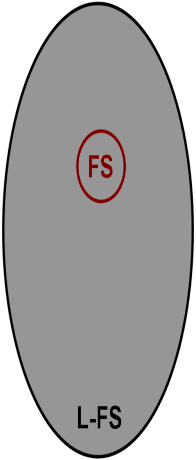
2. There exist situations where experts do not have all the necessary information to build the ideal membership grades.

The origin of extensions is in the trial of simultaneously representing both levels of imperfect information.

- E.E. Kerre, A first view on the alternatives of fuzzy sets theory, in : B. Reusch, K-H Temme (Eds), Computational intelligence in Theory and Practice, Physica-Verlag, Heidelberg (2001) 55-72

Type-2 fuzzy sets

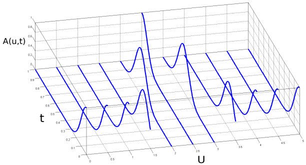
- L. A. Zadeh, Quantitative fuzzy semantics, Inform. Sci. 3 (1971) 159-176



Definition

A type-2 fuzzy set is a mapping:

$$A : U \to FS([0,1])$$



- Type-2 fuzzy sets are a particular case of L-fuzzy sets.
- $T2FS(U) \equiv (FS([0,1]))^U$

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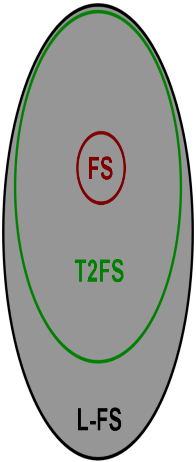

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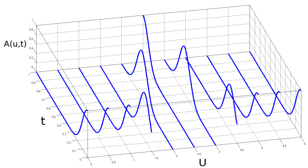

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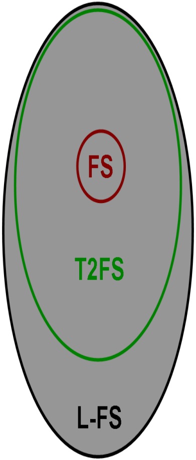
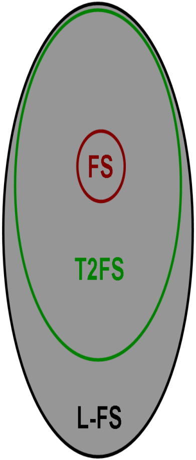


Problems:

1. Notation
   - M. Mizumoto, K. Tanaka, Some properties of fuzzy sets of type 2, Inform. Control, 31, (1976), 312-340
   - J.M. Mendel, R. John, Type-2 Fuzzy Sets Made Simple, IEEE Transactions on Fuzzy Systems 10(2) (2002) 117-127

$$\int_{u \in U} \int_{t \in J_u} A(u,t)/(u,t) \qquad J_u \subset [0,1]$$

where J_u is the primary membership of $u \in U$ and, for each fixed $u = u_0$, the fuzzy set $\int_{t \in J_{u_0}} A(u_0,t)/t$ is the secondary membership of u_0.



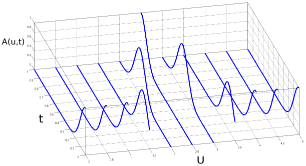
Definition

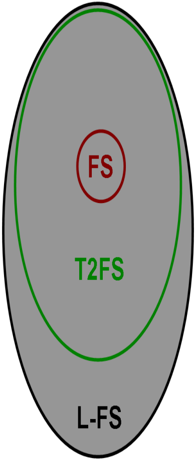
Let $A : U \to FS([0,1])$ *be a type 2 fuzzy set. Then* A *is denoted as*

$$\{(u_i, A(u_i,t)) \mid u_i \in U \ , \ t \in [0,1]\} \ .$$

where $A(u_i,\cdot) : [0,1] \to [0,1]$ *is defined as*

$$A(u_i,t) = A(u_i)(t)$$





2 Structure

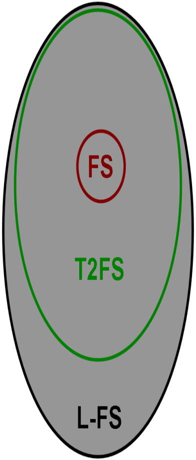
- M. Mizumoto, K. Tanaka, Some properties of fuzzy sets of type 2, Inform. Control, 31, (1976), 312-340
- D. Dubois, H. Prade, Operations in a fuzzy-valued logic, Inform. Control, 43(2), (1979) 224-254

Definition

$$A \cup_{T2} B(u_i) = A(u_i) \cup B(u_i)$$
$$A \cap_{T2} B(u_i) = A(u_i) \cap B(u_i)$$

Proposition

$(T2FS(U), \cup_{T2}, \cap_{T2})$ is a bounded lattice with respect to the order: $A \leq_{T2FS(U)} B$ if and only if $A \cup_{T2} B = B$



Definition

$$A = \{(u_i, A(u_i,t)) \mid u_i \in U \;,\; t \in [0,1]\}$$
$$B = \{(u_i, B(u_i,t)) \mid u_i \in U \;,\; t \in [0,1]\}$$

$\bullet A \sqcap B = \{(u_i, A \sqcap B(u_i,t)) \mid u_i \in U \;,\; t \in [0,1]\}$
$A \sqcap B(u_i,t) = \sup_{min(z,w)=t} min(A(u_i,z), B(u_i,w))$

$\bullet A \sqcup B = \{(u_i, A \sqcup B(u_i,t)) \mid u_i \in U \;,\; t \in [0,1]\}$
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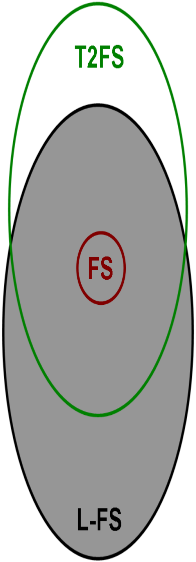
$(T2FS(U), \sqcup, \sqcap)$ **is NOT a lattice**

Type-2 fuzzy sets



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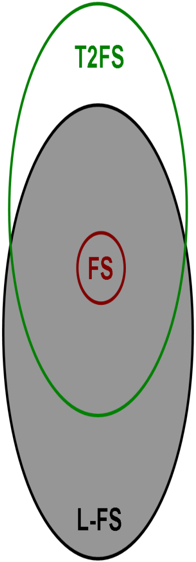
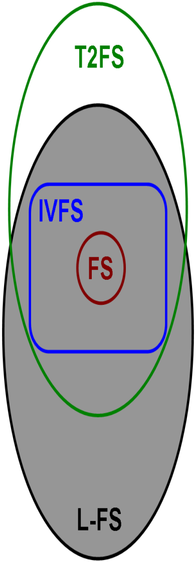


3. Computational efficiency: regression to infinity
4. Applications

   It does not exist yet an application that shows the advantage of using these sets.

   1. **Computing with words**:
      - J.M. Mendel Type-2 fuzzy sets for computing with words Conference Information: IEEE International Conference on Granular Computing, MAY 10-12, 2006 Atlanta, (2006) GA 8-8.
      - J.M. Mendel, Computing with words and its relationships with fuzzistics Information sciences 177(4) (2007) 988-1006
   2. **Perceptual computing**: JM Mendel,
   3. **Control**:
      - H. Hagras, A Hierarchical Type-2 Fuzzy Logic Control Architecture for Autonomous Mobile Robots, IEEE Transactions on Fuzzy Systems 12, (2004) 524-539.

Interval-valued fuzzy sets



1. In 1975 Sambuc: Φ-flou
2. Name of interval-valued fuzzy sets, 80s (Gorzalczany and Turksen)

Definition

An interval-valued fuzzy set is a mapping:

$$A : U \to L([0,1])$$

$A(u_i) = [\underline{A}(u_i), \overline{A}(u_i)]$ denotes the membership grade of u_i to A.

- They are a particular case of L-fuzzy sets
- $L([0,1]) = \{\mathbf{x} = [\underline{x}, \overline{x}] | (\underline{x}, \overline{x}) \in [0,1]^2$ and $\underline{x} \leq \overline{x}\}$

- J.L. Deng, Introduction to grey system theory, Journal of Grey Systems 1 (1989) 1-24

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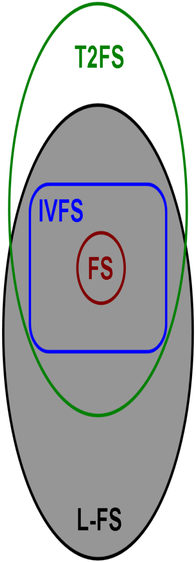

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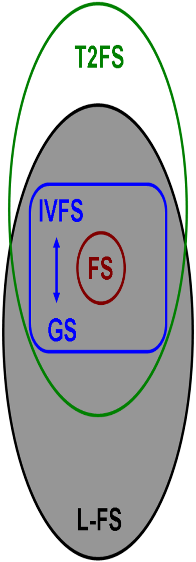

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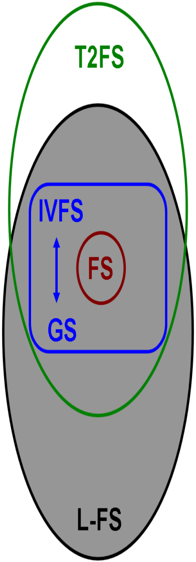
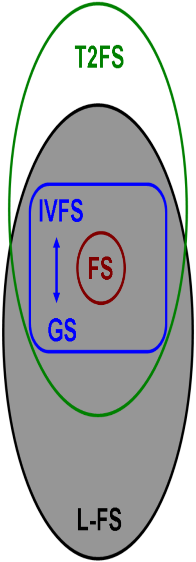


Definition

$$A \cup_{L([0,1])} B(u_i) =$$
$$[max(\underline{A}(u_i), \underline{B}(u_i)), max(\overline{A}(u_i), \overline{B}(u_i))]$$
$$A \cap_{L([0,1])} B(u_i) =$$
$$[min(\underline{A}(u_i), \underline{B}(u_i)), min(\overline{A}(u_i), \overline{B}(u_i))]$$

$(IVFS(U), \cup_{L([0,1])}, \cap_{L([0,1])})$ is a complete lattice



A.- Mathematical interpretation. Theoretical interest.

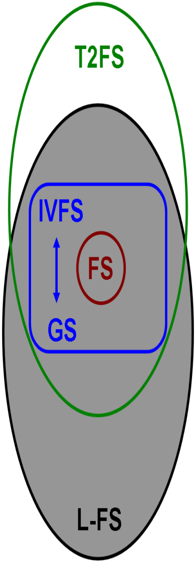
D. Dubois' paradox:

$$min(A(u_i), 1 - A(u_i)) \leq 0{,}5$$

$$min([\underline{A}(u_i), \overline{A}(u_i)], [1 - \overline{A}(u_i), 1 - \underline{A}(u_i)]) \leq ??$$

- H.Bustince, F.Herrera, J.Montero (Eds.), Fuzzy Sets and Their Extensions: Representation Aggregation and Models, Springer, Berlin, 2007.

B.- The expert does not know the exact value of the membership of the element to the fuzzy set. However, the expert knows that this value is bounded by the bounds of the interval-valued membership to the IVFS.



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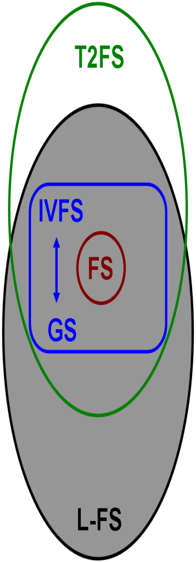

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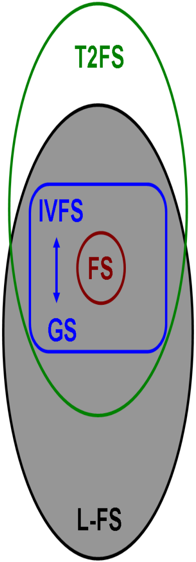

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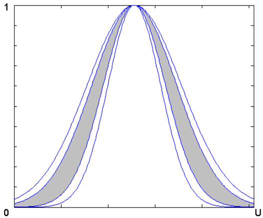
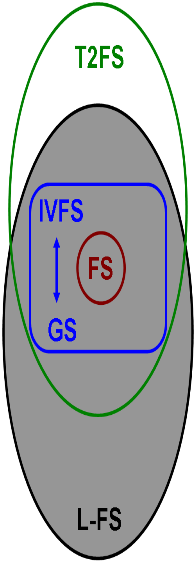


- G. Deschrijver, E.E. Kerre, On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision, Information Sciences 177, (2007) 1860-1866
- J.M. Mendel, Advances in type-2 fuzzy sets and systems, Information Sciences 177, (2007) 84-110

2000, Name:

Interval type-2 fuzzy sets

Interval-valued fuzzy sets and type-2 fuzzy sets



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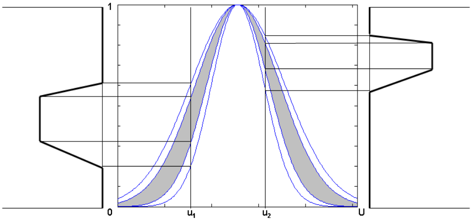

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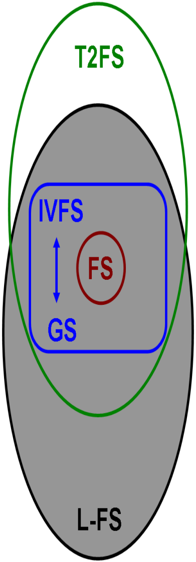
Problems with interval-valued fuzzy sets



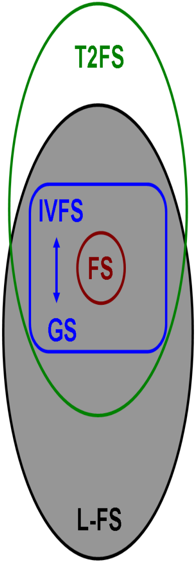
1.- A great number of works are a mere copy of developments already done for fuzzy sets.

2.- Applications start from interval-valued data. It is not possible to compare with fuzzy methods.

3.- Results of the numerical information measures must be interval-valued. Ex: Compatibility grade of Gorzalczany.

4.- Computational efficiency.

5.- Two names: interval-valued fuzzy sets, interval type-2 fuzzy sets.



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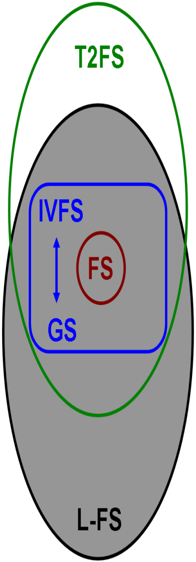
Problems with interval-valued fuzzy sets



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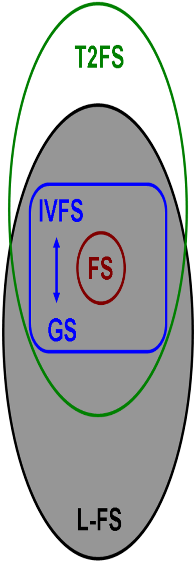

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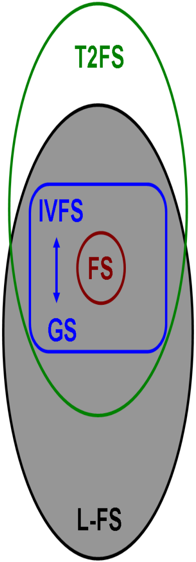

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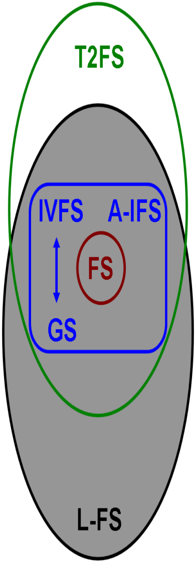

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Definition

An Atanassov's intuitionistic fuzzy set over U *is an expression* A *given by*

$$A = \{(u_i, \mu_A(u_i), \nu_A(u_i))|u_i \in U\}, \text{ where}$$
$$\mu_A : U \longrightarrow [0,1]$$
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such that $0 \leq \mu_A(u_i) + \nu_A(u_i) \leq 1$ *for every* $u_i \in U$

Negation

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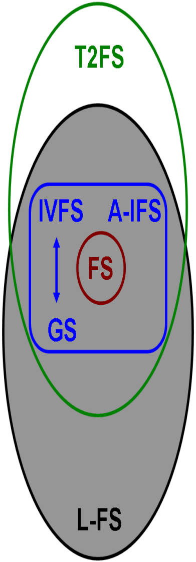

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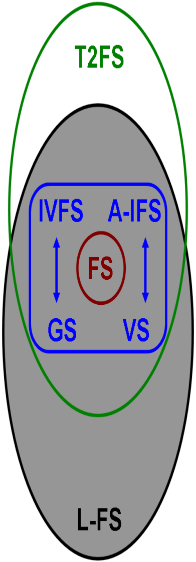

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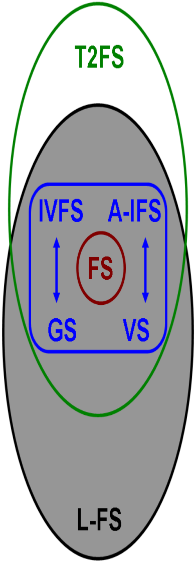

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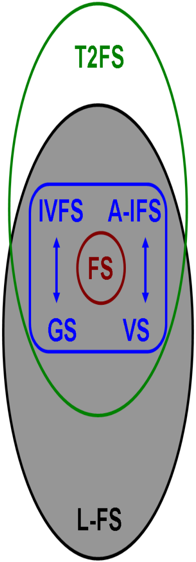
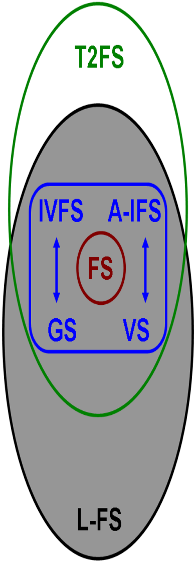


- Atanassov's intuitionistic fuzzy sets are a particular case of L-fuzzy sets.
- $\mathcal{L} = \{(x_1, x_2) | x_1 + x_2 \leq 1\}$

$$A \cup B =$$
$$\{(u_i, max(\mu_A(u_i), \mu_B(u_i)), min(\nu_A(u_i), \nu_B(u_i))) | u_i \in U\}$$
$$A \cap B =$$
$$\{(u_i, min(\mu_A(u_i), \mu_B(u_i)), max(\nu_A(u_i), \nu_B(u_i))) | u_i \in U\}$$

$(A - IFS(U), \cup, \cap)$ is a complete lattice



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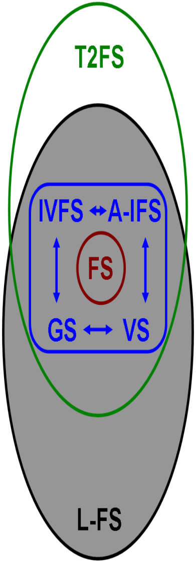
Theorem

The mapping

$$\Phi : IVFS(U) \to A - IFS(U)$$
$$A \to A',$$

where $A' = \{(u_i, \underline{A}(u_i), 1 - \overline{A}(u_i)) | u_i \in U\}$, *is a bijection*



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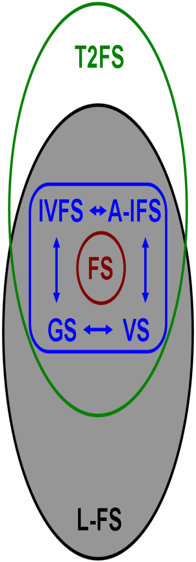
Problems with A-IFSs



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3.- The results of numerical information measures should be Atanassov's intuitionistic pairs.

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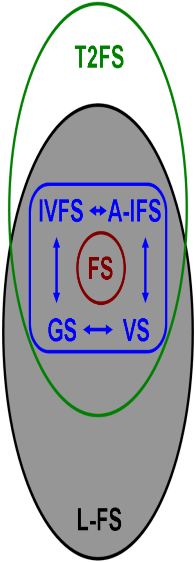

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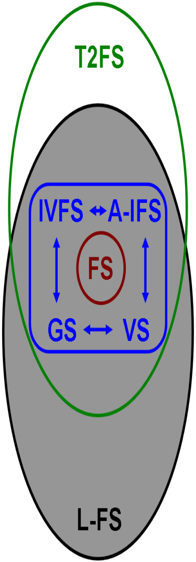

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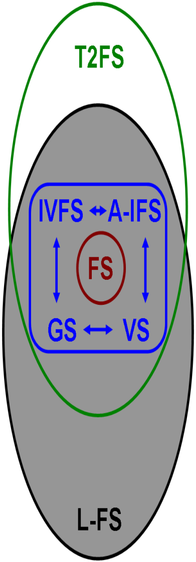

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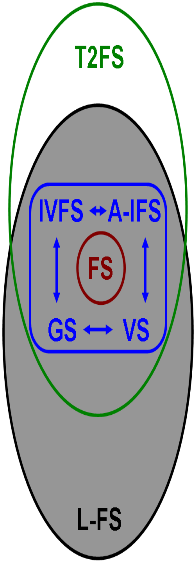


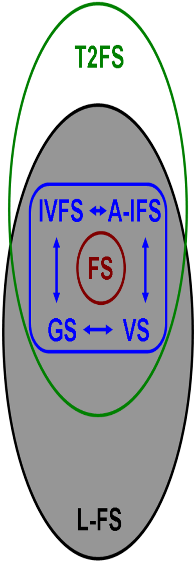
5.- Order

$(\mu_A, \nu_A) \leq (\mu_B, \nu_B)$ if and only if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$
Partial order. It has been extended to a total order in several ways, as

- Z.Xu and R.Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, Int. J. General Syst, 35, (2006) 417-433

$(\mu_A, \nu_A) \leq (\mu_B, \nu_B)$ if and only if
$\mu_A - \nu_A \leq \mu_B - \nu_B$ or
$\mu_A - \nu_A = \mu_B - \nu_B$ and $\mu_A + \nu_A \leq \mu_B + \nu_B$

- The meaning of these total orders is not justified.
- The extensions of usual fuzzy operators to the intuitionistic setting does not preserve in general the monotonicity with respect to this total order.



6.- Name

1. Brouwer's intuitionistic logic
2. Intuitionistic sets by G. Takeuti and S. Titani
   - G. Takeuti, S. Titani, Intuitionistic fuzzy logic and intuitionistic fuzzy set theory, J. Symbolic Logic 49 (1984) 851-866
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   - K. Atanassov, Answer to D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade's paper Terminological difficulties in fuzzy set theory, the case of Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 156(3), (2005) 496-499
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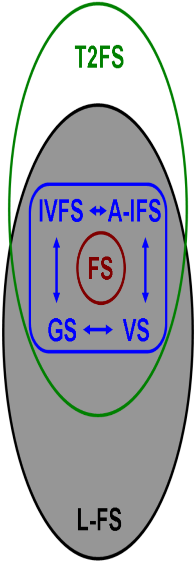

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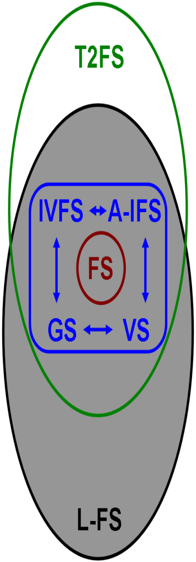

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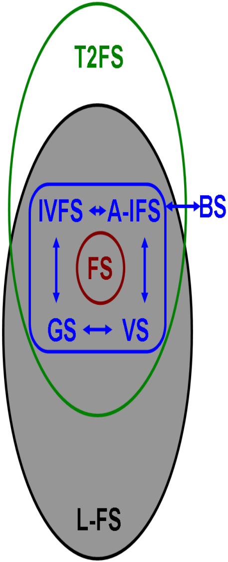


W. R. Zhang: NPN Fuzzy Sets and NPN Qualitative Algebra: A Computational Framework for Bipolar Cognitive Modeling and Multiagerit Decision Analysis. *IEEE Transactions on Systems, Man, and Cybernetics-part B: Cybernetics* 26(4) (1996), 561-574.

Definition

A bipolar valued set on U *is an object*

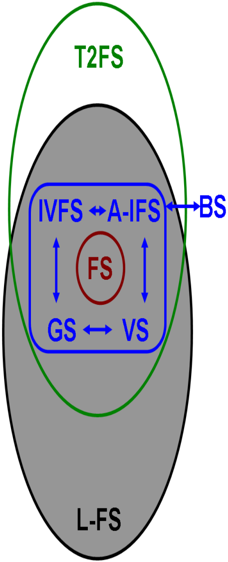
$$A = \{(u_i, \varphi^+(u_i), \varphi^-(u_i)) | u_i \in U\}$$

with

$$\varphi^+ : U \to [0, 1]$$

$$\varphi^- : U \to [-1, 0]$$

- J.T. Cacioppo, W.L. Gardner, G.G. Bernston: Beyond bipolar conceptualizations and measures: The case of attitudes and evaluative space. *Personality and Social Psychology Review* 1(1) (1997), 3-25
- F. Smarandache: A unifying field in logics: neutrosophic logic, Multiple-Valued Logic 8(3) (2002), 385-438
- D. Dubois, H. Prade: An overview of the asymmetric bipolar representation of positive and negative information in possibility theory, Fuzzy sets and Systems 160(10) (2009) 1355-1366

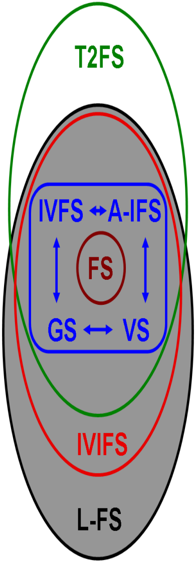


Definition

A symmetric bipolar set on U *is an object*

$$A = \{(u_i, \mu_A(u_i), \nu_A(u_i)) | u_i \in U\}, \text{ with}$$
$$\mu_A : U \to [0,1]$$
$$\nu_A : U \to [0,1]$$

1. No restrictions. Paradox of D. Dubois
2. Atanassov's intuitionistic fuzzy sets are a particular case.
3. Widely used in Psychology



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Definition

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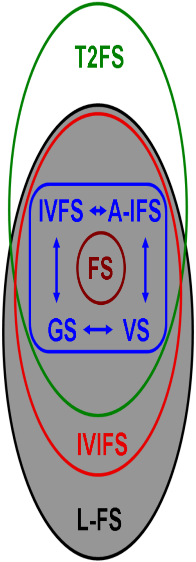
$$M_A : U \longrightarrow L([0,1])$$
$$N_A : U \longrightarrow L([0,1])$$

such that $0 \leq \overline{M_A}(u_i) + \overline{N_A}(u_i) \leq 1$ *for every* $u_i \in U$

- Atanassov's interval-valued intuitionistic fuzzy sets are a particular case of L-fuzzy sets.
- $L([0,1]) = \{\mathbf{x} = [\underline{x}, \overline{x}] | (\underline{x}, \overline{x}) \in [0,1]^2 \text{ and } \underline{x} \leq \overline{x}\}$



Definition

$$A \cup B(u_i) =$$
$$([max(\underline{M_A}(u_i), \underline{M_B}(u_i)), max(\overline{M_A}(u_i), \overline{M_B}(u_i))]$$
$$[min(\underline{N_A}(u_i), \underline{N_B}(u_i)), min(\overline{N_A}(u_i), \overline{N_B}(u_i))])$$
$$A \cap B(u_i) =$$
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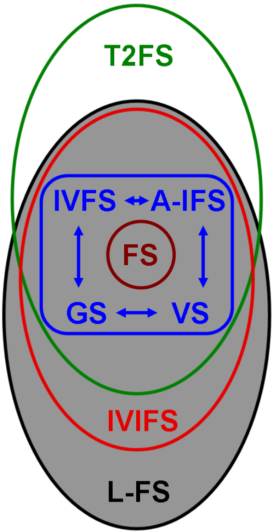
$$[max(\underline{N_A}(u_i), \underline{N_B}(u_i)), max(\overline{N_A}(u_i), \overline{N_B}(u_i))])$$

$(IVIAFS(U), \cup, \cap)$ is a complete lattice

Other extensions



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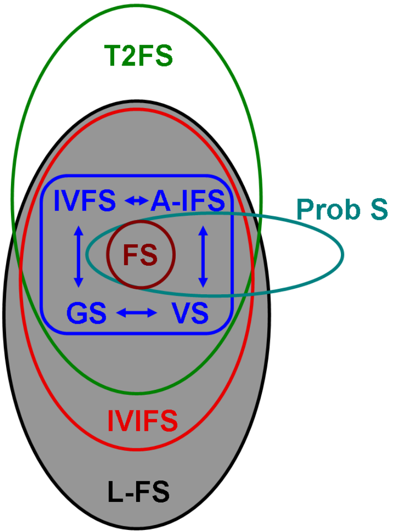

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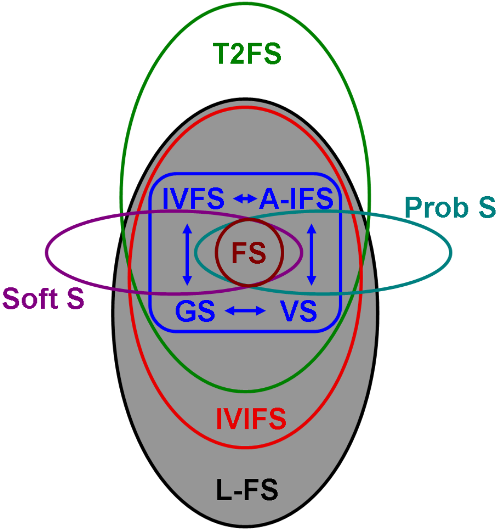
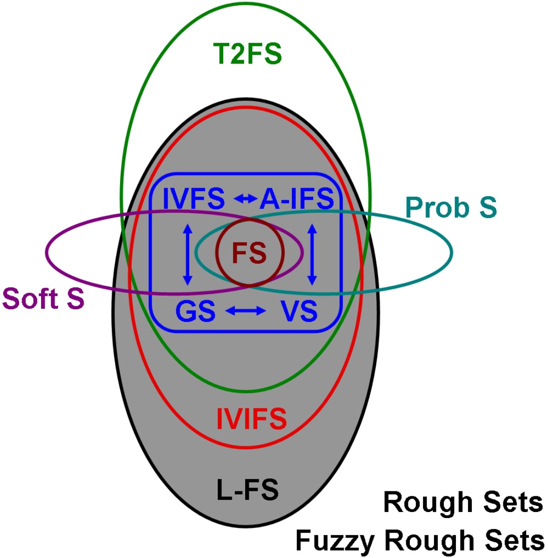



T2FS
IVFS ↔ A-IFS
Prob S
FS
Soft S
GS ↔ VS
IVIFS
L-FS
Rough Sets
Fuzzy Rough Sets

When should we use extensions?

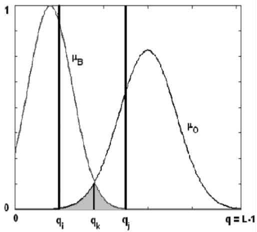
Rule 1:

We must not impose the use of fuzzy sets or of any extension for problems dealing with imperfect information.

Given a problem, if we decide to use fuzzy sets to solve it, it may happen:

1. Acceptable results, we keep using fuzzy sets.
2. Non-acceptable results since experts have problems to build the fuzzy sets; then, we must use, for this specific problem, one extension.
3. Non-acceptable results since fuzzy techniques are not suitable.

When should we use extensions?



When should we use extensions?

Rule 2:

The choice of the extension depends on the nature of the problem

- Interval-valued fuzzy sets
- Atanassov's intuitionistic fuzzy sets

Rule 3:

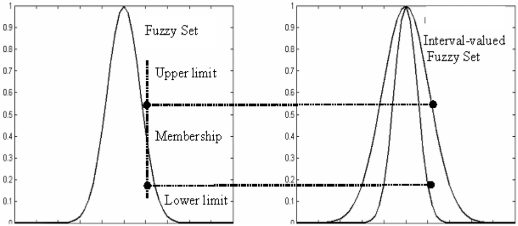
If an extension is used, results must always be compared with those obtained without using fuzzy sets.

Goal

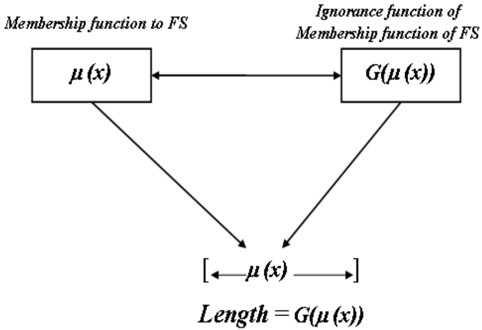
To improve the results obtained with fuzzy sets

- It will be necessary to find a way to **compare** to the results obtained with fuzzy sets.
- To do so, we will build the extensions from the fuzzy sets given by the experts:
  1. From a single fuzzy set given by an expert:
     - Tizhoosh's method
     - Ignorance functions
  2. From several fuzzy sets provided by several experts

Construction: Tizhoosh's method



- $M_{A_t}(q) = [\mu_{A_t}^{\alpha}(q), \mu_{A_t}^{\frac{1}{\alpha}}(q)]$, with $\alpha > 1$
- length does not a specific meaning



- $G \equiv$ ignorance (lack of information) associated to the membership grade provided by the expert.
- length is proportional to that ignorance.

Construction: Ignorance functions

Definition

An ignorance function is a continuous function $G_i : [0,1]^2 \to [0,1]$ *such that:*

$G_i1)$ $G_i(x,y) = G_i(y,x)$ *for every* $x, y \in [0,1]$;

$G_i2)$ $G_i(x,y) = 0$ *if and only if* $x = 1$ *or* $y = 1$;

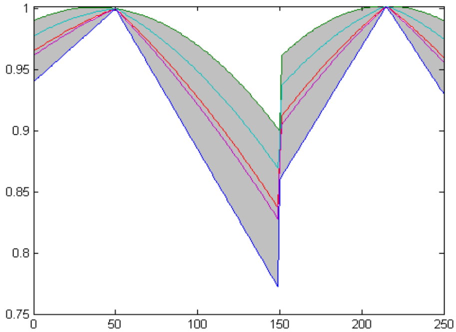
$G_i3)$ *If* $x = 0{,}5$ *e* $y = 0{,}5$, *then* $G_i(x,y) = 1$;

$G_i4)$ G_i *is decreasing in* $[0{,}5, 1]^2$;

$G_i5)$ G_i *is increasing in* $[0, 0{,}5]^2$.

- H. Bustince, M. Pagola, E. Barrenechea, J. Fernandez, P. Melo-Pinto, P. Couto, H.R. Tizhoosh, J. Montero, Ignorance functions. An application to the calculation of the threshold in prostate ultrasound images, Fuzzy Sets and Systems, 161(1) 2010, 20-36

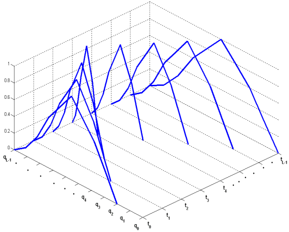
Construction from several fuzzy sets

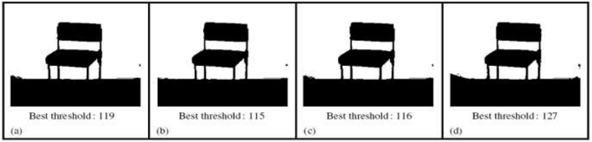


- Advantage: K_α operators, recovering of fuzzy
- length does not have a specific meaning

Applications: image

- 1979: Otsu's algorithm
- 1995: Huang-Wang's algorithm

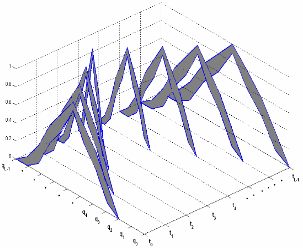




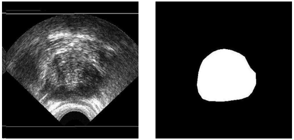
Referential set $\{t = 0, \cdots, L-1\} \Longleftrightarrow [0, 1]$

Applications: image

- 2005: H.R. Tizhoosh, Image thresholding using type-2 fuzzy sets, Pattern Recognition 38, 2363-2372
- 2008: I. K. Vlachos, G. D. Sergiadis, Comment on: Image thresholding using type II fuzzy sets, Pattern Recognition 41 (5) 1810-1811
- 2010: H. Bustince, E. Barrenechea, M. Pagola, J. Fernandez, J. Sanz, Comment on: Image thresholding using type II fuzzy sets. Importance of this method, Pattern Recognition, 43(9), 3188-3192

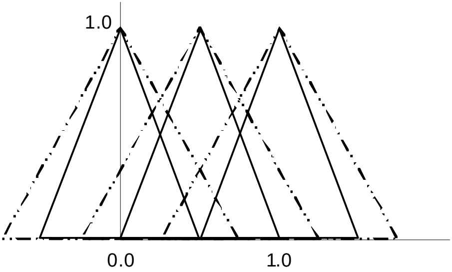


- $G_i(x, y) = 2min(1 - x, 1 - y)$
- To minimize ignorance (length)
- H. Bustince, M. Pagola, E. Barrenechea, J. Fernandez, P. Melo-Pinto, P. Couto, H.R. Tizhoosh, J. Montero, Ignorance functions. An application to the calculation of the threshold in prostate ultrasound images, Fuzzy Sets and Systems, 161(1) 2010, 20-36.



Applications: Classification

- Classification systems based on fuzzy rules with IVFSs.
  - We generate the knowledge base (KB) by already known rule generating algorithms.
  - From this KB we include the IVFSs.
  - We modify the fuzzy reasoning method.
  - We carry on a genetic post-processing: length of the support of the upper bound.
  - We statistically improve the base models.



Conclusions

1. We must never force the use of one extension instead of fuzzy sets or vice-versa.
2. Most extensions arise to model the imprecision in the building of fuzzy sets.
3. There are important problems of interpretation, names,...
4. If we use one extension, we must also use the specific features of that extension.
5. The use of extensions for a given application is only justified if it improves the results obtained with fuzzy sets.