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Fuzzy identities and fuzzy equational classes

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Abstract I

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Dealing with fuzzy relations on fuzzy sets, we investigate fuzzy congruences on fuzzy subalgebras of a crisp algebra. The set of membership values for all fuzzy structures is a naturally ordered commutative, idempotent monoid, which in some cases possesses additional properties (e.g., it is a complete lattice or a complete Heyting algebra).

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Dealing with fuzzy relations on fuzzy sets, we investigate fuzzy congruences on fuzzy subalgebras of a crisp algebra. The set of membership values for all fuzzy structures is a naturally ordered commutative, idempotent monoid, which in some cases possesses additional properties (e.g., it is a complete lattice or a complete Heyting algebra).
Special fuzzy congruences are compatible fuzzy equalities and these are introduced in order to replace the ordinary crisp equality in dealing with fuzzy identities. A consequence of our approach (fuzzy relations on fuzzy structures) is that these relations apart from being symmetric, transitive and compatible with operations, are weakly reflexive (instead of being reflexive in the classical sense).

Fuzzy identities and fuzzy equational classes

Abstract cont.

II

II

Consequently, we define fuzzy identities on a fuzzy subalgebra, as formulas in which terms in the language of an algebra are related by compatible fuzzy equalities. A fuzzy identity may be satisfied by a fuzzy subalgebra (with respect to some fuzzy equality), while the underlying crisp algebra need not satisfy the analogue crisp identity. Among other properties, we prove that if a fuzzy subalgebra of an algebra satisfies a fuzzy identity with respect to some fuzzy equality, then there is a least fuzzy equality such that the corresponding fuzzy identity holds on the same fuzzy subalgebra. This could be understood as a 'degree of fuzzinees' of a property described by an identity (e.g., commutativity, associativity etc.).

Fuzzy identities and fuzzy equational classes

Abstract cont.

III

III

Next we introduce and investigate fuzzy equational classes. These are defined with respect to a set of fuzzy identities, and consist of fuzzy algebras of the same type, fulfilling all fuzzy identities in the given set. Fuzzy algebras are by definition fuzzy subalgebras of crisp algebras, equipped with a compatible fuzzy equality. In this fuzzy framework we introduce basic notions of universal algebra: fuzzy homomorphisms (H), fuzzy subalgebras (S), and fuzzy direct products (P). Our main result is that every fuzzy equational class is closed under these three constructions (H,S and P), hence forming a fuzzy variety.

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Terms in the language $\mathcal{L}$ are usual regular expressions constructed by the variables and operational symbols.
An **identity** in $\mathcal{L}$ is a formula $t_1 = t_2$, where t_1, t_2 are terms in the same language.

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If $\mathcal{A}$ and $\mathcal{B}$ are algebras of the same type, then the function $h : A \to B$ compatible with fundamental operations is a **homomorphism** of $\mathcal{A}$ into $\mathcal{B}$.

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For a family $\{\mathcal{A}_i \mid i \in I\}$ of algebras of the same type, $\Pi_{i\in I}\mathcal{A}_i$ is their **direct product**, an algebra of the same type with operations defined componentwise.

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Theorem (Birkhoff)

The following are equivalent for a class $\mathcal{V}$ of algebras of the same type:
(i) $\mathcal{V}$ is an equational class.
(ii) $\mathcal{V}$ is a variety.

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A/ρ is a homomorphic image of $\mathcal{A}$ under $x \mapsto [x]_\rho$, where $[x]_\rho$ is the congruence class to which x belongs.

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Lemma

The following are obvious properties of ordered structure $(L, \leqslant)$:

- *Infimum (meet) coincides with $\otimes$.*
- *The unit 1 is the top element wit respect to $\leqslant$.*
- $\inf L$ *is the bottom element in $(L, \leqslant)$, we denote it by 0.*
- $(L, \leqslant)$ *is a complete lattice.*

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If $\mu : A \to L$ is a fuzzy set on a nonempty set A, then a fuzzy relation $\rho : A^2 \to L$ on A is said to be a **fuzzy relation on** μ if for all $x, y \in A$

$$\rho(x, y) \leqslant \mu(x) \otimes \mu(y).$$

A fuzzy relation ρ on a fuzzy set μ is **reflexive** if for all $x, y \in A$,

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Lemma

If ρ is a reflexive fuzzy relation on a fuzzy set μ on A, then for every $x, y \in A$,

$$\rho(x, x) \geqslant \rho(x, y) \ \ \textit{and} \ \ \rho(x, x) \geqslant \rho(y, x).$$

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A reflexive, symmetric and transitive relation ρ on a fuzzy set μ is a **fuzzy equivalence** on μ.

A fuzzy equivalence relation ρ on μ, fulfilling for all $x, y \in A$, $x \neq y$,:

if $\rho(x, x) \neq 0$, then $\rho(x, x) > \rho(x, y)$ and $\rho(x, x) > \rho(y, x)$,

is called a **fuzzy equality** relation on a fuzzy set μ.

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For any operation f from F with arity greater than 0, $f : A^n \to A, n \in \mathbb{N}$, and all $x_1, \ldots, x_n \in A$, we have that

$$\bigotimes_{i=1}^{n} \mu(x_i) \leqslant \mu(f(x_1, \ldots, x_n)).$$

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For a nullary operation (constant) $c \in F$, we require that

$$\mu(c) = 1,$$

where 1 is the greatest (the top) element in L.

A fuzzy relation $\rho : A^2 \to L$ on a fuzzy subalgebra $\mu : A \to L$ of $\mathcal{A} = (A, F)$ is said to be **compatible** with the operations, if for every n-ary operation $f \in F$ and for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$

$$\rho(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \geqslant \bigotimes_{i=1}^{n} \rho(x_i, y_i).$$

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Obviously, particular fuzzy congruences on μ are compatible fuzzy equalities on this fuzzy subalgebra.

Fuzzy identity

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A fuzzy **identity** of the type (F, σ) over a set of variables X is the expression $E(t_1, t_2)$, where $t_1(x_1, \ldots, x_n), t_2(x_1, \ldots, x_n)$, briefly t_1, t_2 belong to the set $T(X)$ of terms over X and both have at most n variables.

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A fuzzy subalgebra μ of an algebra $\mathcal{A} = (A, F)$ **satisfies** a fuzzy identity $E(t_1, t_2)$ (this identity **holds** on μ) with respect to fuzzy equality E_i on $\mathcal{A}$, if for all $x_1, \ldots, x_n \in A$

$$\bigotimes_{i=1}^{n} \mu(x_i) \leqslant E_i(t_1, t_2).$$

Theorem

Let μ be a fuzzy subalgebra of $\mathcal{A}$, such that there is a fuzzy equality E on μ so that μ satisfies identity $E(f,g)$ for terms f,g in the language of $\mathcal{A}$. Then there is the least fuzzy equality on μ, denoted by $E_{\mu(f,g)}$, such that μ satisfies $E_{\mu(f,g)}(f,g)$.

Fuzzy algebra

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Let $\mathcal{A} = (A, F_A)$ be an algebra, let $\mu_A : A \to L$ be a fuzzy subalgebra of $\mathcal{A}$ and $E_A : A^2 \to L$ a compatible fuzzy equality on μ_A.

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Then, $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ is a **fuzzy algebra** of the type (F, σ).

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Let $\mathcal{A} = (A, F_A)$ be an algebra, let $\mu_A : A \to L$ be a fuzzy subalgebra of $\mathcal{A}$ and $E_A : A^2 \to L$ a compatible fuzzy equality on μ_A.

Then, $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ is a **fuzzy algebra** of the type (F, σ).

In other words, a *fuzzy algebra is a fuzzy L-valued subalgebra of a given crisp algebra, endowed with a compatible fuzzy equality.*

A fuzzy algebra $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ **satisfies** a fuzzy identity $E(t_1, t_2)$ (this identity **holds** on $\bar{\mathcal{A}}$), if for all $x_1, \ldots, x_n \in A$

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Equational class

Let Σ be a set of fuzzy identities of the type (F, σ) and let L be a fixed complete lattice. Then all fuzzy algebras $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ of this type satisfying all identities in Σ form an **equational class** $\mathfrak{M}$ of fuzzy algebras.

Operator S

Operator S

Theorem

Let $\bar{A} = (\mathcal{A}, \mu_A, E_A, L)$ *be a fuzzy algebra and* $\mu_B : A \to L$ *a fuzzy subalgebra of* $\mathcal{A}$*, fulfilling the following conditions:*

1. $\mu_B(x) \leqslant \mu_A(x)$ *for all* $x \in A$.
2. *If* x *and* y *are distinct elements of* A *and* $\mu_B(x) > 0$*, then* $E_A(x, y) < \mu_B(x)$.
3. $\mu_B(c) = \mu_A(c)$ *, for any constant* c *in the language.*

Then, a fuzzy relation E_B *on* μ_B *given by*

$$E_B(x, y) := E_A(x, y) \otimes \mu_B(x) \otimes \mu_B(y),$$

is a compatible fuzzy equality on μ_B.

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ be a fuzzy algebra and $\mu_B : A \to L$ a fuzzy subalgebra of $\mathcal{A}$, fulfilling the following:

1. $\mu_B(x) \leqslant \mu_A(x)$ for all $x \in A$.
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3. $\mu_B(c) = \mu_A(c)$, for any constant c.
4. $E_B(x, y) := E_A(x, y) \otimes \mu_B(x) \otimes \mu_B(y)$.

Then we say that the fuzzy algebra $\bar{\mathcal{B}} = (\mathcal{A}, \mu_B, E_B, L)$ is a (**fuzzy**) **subalgebra** of the fuzzy algebra $\bar{\mathcal{A}}$.

Theorem

Let $\mathfrak{M}$ be an equational class of fuzzy algebras and let $\bar{\mathcal{A}} \in \mathfrak{M}$ where $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$. If $\bar{\mathcal{B}} = (\mathcal{A}, \mu_B, E_B, L)$ is a fuzzy subalgebra of $\bar{\mathcal{A}}$, then also $\bar{\mathcal{B}} \in \mathfrak{M}$.

Operator H

Operator H

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ and $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ be fuzzy algebras of the same type. We say that $f : A \to B$ is a **fuzzy mapping** of $\bar{\mathcal{A}}$ into $\bar{\mathcal{B}}$ if the following conditions hold:
1. $(\forall a \in A)\ \mu_B(f(a)) \geqslant \mu_A(a)$
2. Let $t_1(x_1, ..., x_n)$, $t_2(x_1, ..., x_n)$ be terms in the language of $\mathcal{A}$, let t_1^A, t_2^A be the corresponding term operations and $a_1, ..., a_n$ elements from A.

$$\text{If}\quad E_A(t_1^A(a_1, ..., a_n), t_2^A(a_1, ..., a_n)) \geqslant \bigotimes_{i=1}^{n} \mu_A(a_i),$$

$$\text{then}\quad E_B(f(t_1^A(a_1, ..., a_n)), f(t_2^A(a_1, ..., a_n))) \geqslant$$

$$\mu_B(f(t_1^A(a_1, ..., a_n))) \otimes \mu_B(f(t_2^A(a_1, ..., a_n))).$$

Theorem

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$, $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ *and* $\bar{\mathcal{C}} = (\mathcal{C}, \mu_C, E_C, L)$ *be fuzzy algebras of the same type and* $f : A \to B$, $g : B \to C$ *fuzzy mappings. Then also their composition* $f \circ g : A \to C$ *is a fuzzy mapping.*

Theorem

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$, $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ and $\bar{\mathcal{C}} = (\mathcal{C}, \mu_C, E_C, L)$ be fuzzy algebras of the same type and $f : A \to B$, $g : B \to C$ fuzzy mappings. Then also their composition $f \circ g : A \to C$ is a fuzzy mapping.

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ and $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ be fuzzy algebras of the same type. Then a fuzzy mapping $h : A \to B$ is a **fuzzy homomorphism** of $\bar{\mathcal{A}}$ into $\bar{\mathcal{B}}$ if the following holds:

1. For each n-ary operation f_A and for all $a_1, ..., a_n \in A$, $h(f_A(a_1, ..., a_n)) = f_B(h(a_1), ..., h(a_n))$.
2. $h(c_A) = c_B$, for every nullary operation c in the language, c_A and c_B being the corresponding constants in $\mathcal{A}$ and $\mathcal{B}$ respectively.

Proposition

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ *be a fuzzy algebra and* $\mathcal{B} = (B, F_B)$ *a crisp subalgebra of* $\mathcal{A}$*. If*

$$\mu_B(x) := \begin{cases} \mu_A(x), & x \in B \\ 0, & else \end{cases} \quad \textit{and}$$

$$E_B(x, y) := E_A(x, y) \otimes \mu_B(x) \otimes \mu_B(y),$$

then $\bar{\mathcal{B}} = (\mathcal{A}, \mu_B, E_B, L)$ *is a fuzzy subalgebra of* $\bar{\mathcal{A}}$*.*

Theorem

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ *and* $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ *be fuzzy algebras and* $h : A \to B$ *a fuzzy homomorphism. Define* $\bar{\mathcal{D}} = (\mathcal{B}, \mu_D, E_D, L)$, *where*

$$\mu_D(d) := \begin{cases} \mu_B(d), & d \in D = h(A) \\ 0, & \textit{otherwise} \end{cases}$$

and

$$E_D(x, y) = E_B(x, y) \otimes \mu_D(x) \otimes \mu_D(y).$$

Then, $\bar{\mathcal{D}}$ *is a fuzzy subalgebra of the fuzzy algebra* $\bar{\mathcal{B}}$.

Theorem

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ and $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ be fuzzy algebras and $h : A \to B$ a fuzzy homomorphism. If $F(x_1, ..., x_n)$ is a term in the same language and F^A, F^B the corresponding term operations in $\mathcal{A}$ and $\mathcal{B}$ respectively, then h is a fuzzy homomorphism f the fuzzy algebra $(\bar{\mathcal{A}}, F^A)$ into the fuzzy algebra $(\bar{\mathcal{B}}, F^B)$.

Theorem

Let $\mathfrak{M}$ be an equational class of fuzzy algebras. If $\bar{\mathcal{A}} \in \mathfrak{M}$ and $\bar{\mathcal{D}}$ is a homomorphic image of $\bar{\mathcal{A}}$, then also $\bar{\mathcal{D}} \in \mathfrak{M}$.

Operator P

Operator P

Theorem

Let $\{\bar{A}_i = (\mathcal{A}_i, \mu_i, E_{A_i}) \mid i \in I\}$ *be a family of fuzzy algebras of the same type,* $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$ *the direct product of algebras* $\mathcal{A}_i$ *and let the following holds for all* $g_1, g_2 \in \prod_{i\in I} \mathcal{A}_i$*:*

If $g_1 \neq g_2$ *and* $\bigotimes_{i\in I} \mu_i(g_1(i)) \neq 0$*, then*

$$\bigotimes_{i\in I} E_{A_i}(g_1(i), g_2(i)) \neq \bigotimes_{i\in I} \mu_i(g_1(i)).$$

Then the following holds: If

1. $\mu(g) := \bigotimes_{i\in I} \mu_i(g(i)),\ g \in \prod_{i\in I} A_i$ *and*
2. $E_A(g_1, g_2) := \bigotimes_{i\in I} E_{A_i}(g_1(i), g_2(i));\ g_1, g_2 \in \prod_{i\in I} A_i,$

then $\bar{A} = \prod_{i\in I} \bar{A}_i := (\mathcal{A}, \mu, E_A, L)$ *is a fuzzy algebra.*

Then the following holds: If

1. $\mu(g) := \bigotimes_{i \in I} \mu_i(g(i)), \; g \in \prod_{i \in I} A_i$ *and*
2. $E_A(g_1, g_2) := \bigotimes_{i \in I} E_{A_i}(g_1(i), g_2(i)); \; g_1, g_2 \in \prod_{i \in I} A_i,$

then $\bar{\mathcal{A}} = \prod_{i \in I} \bar{\mathcal{A}}_i := (\mathcal{A}, \mu, E_A, L)$ *is a fuzzy algebra.*

The above fuzzy algebra $\bar{\mathcal{A}} := (\mathcal{A}, \mu, E_A, L)$ is the **direct product** of fuzzy algebras $\bar{\mathcal{A}}_i, i \in I$.

Theorem

If a fuzzy identity $E(u(x_1, \ldots, x_n), v(x_1, \ldots, x_n))$ *holds in all fuzzy algebras* $\bar{\mathcal{A}}_i$, $i \in I$ *of a fixed type, then also this fuzzy identity holds in their product* $\bar{\mathcal{A}} = \prod_{i \in I} \bar{\mathcal{A}}_i$.

Theorem

Let $\mathfrak{M}$ be an equational class of fuzzy algebras. Then the following hold:

1. *If $\bar{\mathcal{A}} \in \mathfrak{M}$, and $\bar{\mathcal{B}}$ is a fuzzy subalgebra of $\bar{\mathcal{A}}$, then $\bar{\mathcal{B}} \in \mathfrak{M}$.*
2. *If $\bar{\mathcal{A}} \in \mathfrak{M}$, and $\bar{\mathcal{D}}$ is a homomorphic image of $\bar{\mathcal{A}}$, then $\bar{\mathcal{D}} \in \mathfrak{M}$.*
3. *If for every $i \in I, f\,\bar{\mathcal{A}}_i$ belongs to $\mathfrak{M}$, then also $\prod_{i \in I} \bar{\mathcal{A}}_i \in \mathfrak{M}$.*

Thank you for your attention!