Fuzzy identities and fuzzy equational classes

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Abstract

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Dealing with fuzzy relations on fuzzy sets, we investigate fuzzy congruences on fuzzy subalgebras of a crisp algebra. The set of membership values for all fuzzy structures is a naturally ordered commutative, idempotent monoid, which in some cases possesses additional properties (e.g., it is a complete lattice or a complete Heyting algebra).

Special fuzzy congruences are compatible fuzzy equalities and these are introduced in order to replace the ordinary crisp equality in dealing with fuzzy identities. A consequence of our approach (fuzzy relations on fuzzy structures) is that these relations apart from being symmetric, transitive and compatible with operations, are weakly reflexive (instead of being reflexive in the classical sense).
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Dealing with fuzzy relations on fuzzy sets, we investigate fuzzy congruences on fuzzy subalgebras of a crisp algebra. The set of membership values for all fuzzy structures is a naturally ordered commutative, idempotent monoid, which in some cases possesses additional properties (e.g., it is a complete lattice or a complete Heyting algebra).

Special fuzzy congruences are compatible fuzzy equalities and these are introduced in order to replace the ordinary crisp equality in dealing with fuzzy identities. A consequence of our approach (fuzzy relations on fuzzy structures) is that these relations apart from being symmetric, transitive and compatible with operations, are weakly reflexive (instead of being reflexive in the classical sense).
Consequently, we define fuzzy identities on a fuzzy subalgebra, as formulas in which terms in the language of an algebra are related by compatible fuzzy equalities. A fuzzy identity may be satisfied by a fuzzy subalgebra (with respect to some fuzzy equality), while the underlying crisp algebra need not satisfy the analogue crisp identity. Among other properties, we prove that if a fuzzy subalgebra of an algebra satisfies a fuzzy identity with respect to some fuzzy equality, then there is a least fuzzy equality such that the corresponding fuzzy identity holds on the same fuzzy subalgebra. This could be understood as a 'degree of fuzziness' of a property described by an identity (e.g., commutativity, associativity etc.).
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Next we introduce and investigate fuzzy equational classes. These are defined with respect to a set of fuzzy identities, and consist of fuzzy algebras of the same type, fulfilling all fuzzy identities in the given set. Fuzzy algebras are by definition fuzzy subalgebras of crisp algebras, equipped with a compatible fuzzy equality. In this fuzzy framework we introduce basic notions of universal algebra: fuzzy homomorphisms (H), fuzzy subalgebras (S), and fuzzy direct products (P). Our main result is that every fuzzy equational class is closed under these three constructions (H, S and P), hence forming a fuzzy variety.
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Preliminaries: universal algebra

Universal algebra notions
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A **language** or a **type** \( \mathcal{L} \) is a set \( \mathcal{F} \) of functional symbols, together with a set of natural numbers with zero (arities) associated to these symbols.
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**Terms** in the language $\mathcal{L}$ are usual regular expressions constructed by the variables and operational symbols.

An **identity** in $\mathcal{L}$ is a formula $t_1 = t_2$, where $t_1, t_2$ are terms in the same language.
A class of algebras of the same type, fulfilling a set of identities is an **equational class**.
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If $\mathcal{A}$ and $\mathcal{B}$ are algebras of the same type, then the function $h: \mathcal{A} \rightarrow \mathcal{B}$ compatible with fundamental operations is a **homomorphism** of $\mathcal{A}$ into $\mathcal{B}$.
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The set of images $h(A)$ under $h$ is a subalgebra of $\mathcal{B}$, a **homomorphic image** of $\mathcal{A}$.
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For a family $\{\mathcal{A}_i | i \in I\}$ of algebras of the same type, $\prod_{i \in I} \mathcal{A}_i$ is their **direct product**, an algebra of the same type with operations defined componentwise.
A class of algebras of the same type is a **variety** if it is closed under subalgebras, homomorphic images and direct products.
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**Theorem (Birkhoff)**

*The following are equivalent for a class $\mathcal{V}$ of algebras of the same type:*

(i) $\mathcal{V}$ is an equational class.

(ii) $\mathcal{V}$ is a variety.
An equivalence relation \( \rho \) on \( A \) which is compatible with respect to all fundamental operations is a **congruence** relation on \( A \).
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If $\rho$ is a congruence on $A$, then $A/\rho$ is the **quotient algebra**, where the underlying set consists of congruence classes under $\rho$, and operations are performed over representatives.
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If $\rho$ is a congruence on $A$, then $A/\rho$ is the **quotient algebra**, where the underlying set consists of congruence classes under $\rho$, and operations are performed over representatives.

\[ A/\rho \text{ is a homomorphic image of } A \text{ under } x \mapsto [x]_{\rho}, \text{ where } [x]_{\rho} \text{ is the congruence class to which } x \text{ belongs.} \]
Let \((L, \otimes, 1)\) be a commutative, idempotent monoid, naturally ordered by \(x \leq y\) if and only if \(x \otimes y = x\). Let it be complete with respect to \(\leq\).

**Lemma**

The following are obvious properties of ordered structure \((L, \leq)\):

- Infimum (meet) coincides with \(\otimes\).
- The unit \(1\) is the top element with respect to \(\leq\).
- \(\inf L\) is the bottom element in \((L, \leq)\), we denote it by \(0\).
- \((L, \leq)\) is a complete lattice.
Fuzzy identities and fuzzy equational classes

Preliminaries: fuzzy notions

Membership values structure

Let $(L, \otimes, 1)$ be a commutative, idempotent monoid, naturally ordered by
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Consequently, a mapping $\rho : A^2 \to L$ is a **fuzzy (binary) relation** on $A$. 
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If $\mu : A \to L$ is a fuzzy set on a nonempty set $A$, then a fuzzy relation $\rho : A^2 \to L$ on $A$ is said to be a **fuzzy relation on** $\mu$ if for all $x, y \in A$

$$\rho(x, y) \leq \mu(x) \otimes \mu(y).$$
A fuzzy relation \( \rho \) on a fuzzy set \( \mu \) is **reflexive** if for all \( x, y \in A \),

\[
\rho(x, x) = \mu(x).
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A fuzzy relation $\rho$ on a fuzzy set $\mu$ is **reflexive** if for all $x, y \in A$,

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**Lemma**

*If $\rho$ is a reflexive fuzzy relation on a fuzzy set $\mu$ on $A$, then for every $x, y \in A$,*

$$\rho(x, x) \geq \rho(x, y) \text{ and } \rho(x, x) \geq \rho(y, x).$$
A fuzzy relation on a fuzzy set $\mu$ on $A$ is symmetric and transitive if it fulfills these conditions as a fuzzy relation on the crisp domain $A$:
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A reflexive, symmetric and transitive relation $\rho$ on a fuzzy set $\mu$ is a **fuzzy equivalence** on $\mu$. 

B.& V. Budimirović, B. Šešelja, A. Tepavčević
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A reflexive, symmetric and transitive relation $\rho$ on a fuzzy set $\mu$ is a **fuzzy equivalence** on $\mu$.

A fuzzy equivalence relation $\rho$ on $\mu$, fulfilling for all $x, y \in A$, $x \neq y$:

- if $\rho(x, x) \neq 0$, then $\rho(x, x) > \rho(x, y)$ and $\rho(x, x) > \rho(y, x)$,

is called a **fuzzy equality** relation on a fuzzy set $\mu$. 
Let $\mathcal{A} = (A, F)$ be a crisp algebra.
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For any operation $f$ from $F$ with arity greater than 0, $f : A^n \to A$, $n \in \mathbb{N}$, and all $x_1, \ldots, x_n \in A$, we have that

$$\bigotimes_{i=1}^{n} \mu(x_i) \leq \mu(f(x_1, \ldots, x_n)).$$
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For a nullary operation (constant) $c \in F$, we require that

$$\mu(c) = 1,$$

where 1 is the greatest (the top) element in $L$. 
A fuzzy relation $\rho : A^2 \to L$ on a fuzzy subalgebra $\mu : A \to L$ of $A = (A, F)$ is said to be **compatible** with the operations, if for every $n$-ary operation $f \in F$ and for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$

$$\rho(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \geq \bigotimes_{i=1}^{n} \rho(x_i, y_i).$$
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A compatible fuzzy equivalence on a fuzzy subalgebra \( \mu \) of \( A \) is a **fuzzy congruence** on \( \mu \).
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A compatible fuzzy equivalence on a fuzzy subalgebra $\mu$ of $A$ is a **fuzzy congruence** on $\mu$.

Obviously, particular fuzzy congruences on $\mu$ are compatible fuzzy equalities on this fuzzy subalgebra.
Fuzzy identities and fuzzy equational classes

Fuzzy identity

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A fuzzy identity of the type \((F, \sigma)\) over a set of variables \(X\) is the expression \(E(t_1, t_2)\), where \(t_1(x_1, \ldots, x_n), t_2(x_1, \ldots, x_n)\), briefly \(t_1, t_2\) belong to the set \(T(X)\) of terms over \(X\) and both have at most \(n\) variables.
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A fuzzy subalgebra \(\mu\) of an algebra \(\mathcal{A} = (A, F)\) satisfies a fuzzy identity \(E(t_1, t_2)\) (this identity holds on \(\mu\)) with respect to fuzzy equality \(E_i\) on \(\mathcal{A}\), if for all \(x_1, \ldots, x_n \in A\)

\[
\bigotimes_{i=1}^{n} \mu(x_i) \leq E_i(t_1, t_2).
\]
Fuzzy identities and fuzzy equational classes

Fuzzy identity

**Theorem**

Let \( \mu \) be a fuzzy subalgebra of \( A \), such that there is a fuzzy equality \( E \) on \( \mu \) so that \( \mu \) satisfies identity \( E(f, g) \) for terms \( f, g \) in the language of \( A \). Then there is the least fuzzy equality on \( \mu \), denoted by \( E_{\mu}(f, g) \), such that \( \mu \) satisfies \( E_{\mu}(f, g)(f, g) \).
Let $A = (A, F_A)$ be an algebra, let $\mu_A : A \rightarrow L$ be a fuzzy subalgebra of $A$ and $E_A : A^2 \rightarrow L$ a compatible fuzzy equality on $\mu_A$.

Then, $\bar{A} = (A, \mu_A, E_A, L)$ is a fuzzy algebra of the type $(F, \sigma)$.

In other words, a fuzzy algebra is a fuzzy $L$-valued subalgebra of a given crisp algebra, endowed with a compatible fuzzy equality.
Fuzzy algebra

Let $\mathcal{A} = (A, F_\mathcal{A})$ be an algebra, let $\mu_\mathcal{A} : A \to L$ be a fuzzy subalgebra of $\mathcal{A}$ and $E_\mathcal{A} : A^2 \to L$ a compatible fuzzy equality on $\mu_\mathcal{A}$.

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Then, $\tilde{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ is a fuzzy algebra of the type $(F, \sigma)$. 
Fuzzy algebra

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Then, \( \tilde{\mathcal{A}} = (\mathcal{A}, \mu_{A}, E_{A}, L) \) is a **fuzzy algebra** of the type \( (F, \sigma) \).

In other words, a **fuzzy algebra is a fuzzy L-valued subalgebra of a given crisp algebra, endowed with a compatible fuzzy equality.**
A fuzzy algebra $\overline{A} = (A, \mu_A, E_A, L)$ satisfies a fuzzy identity $E(t_1, t_2)$ (this identity holds on $\overline{A}$), if for all $x_1, \ldots, x_n \in A$

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A fuzzy algebra $\bar{A} = (\mathcal{A}, \mu_A, E_A, L)$ satisfies a fuzzy identity $E(t_1, t_2)$ (this identity holds on $\bar{A}$), if for all $x_1, \ldots, x_n \in A$

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Equational class
A fuzzy algebra $\mathcal{A} = (\mathcal{A}, \mu_\mathcal{A}, E_\mathcal{A}, L)$ satisfies a fuzzy identity $E(t_1, t_2)$ (this identity holds on $\mathcal{A}$), if for all $x_1, \ldots, x_n \in A$

$$\bigotimes_{i=1}^{n} \mu_\mathcal{A}(x_i) \leq E_\mathcal{A}(t_1, t_2).$$

**Equational class**

Let $\Sigma$ be a set of fuzzy identities of the type $(F, \sigma)$ and let $L$ be a fixed complete lattice. Then all fuzzy algebras $\mathcal{A} = (\mathcal{A}, \mu_\mathcal{A}, E_\mathcal{A}, L)$ of this type satisfying all identities in $\Sigma$ form an **equational class** $\mathcal{M}$ of fuzzy algebras.
Let \( \bar{A} = (A, \mu_A, E_A, L) \) be a fuzzy algebra and \( \mu_B : A \to L \) a fuzzy subalgebra of \( A \), fulfilling the following conditions:

1. \( \mu_B(x) \leq \mu_A(x) \) for all \( x \in A \).
2. If \( x \) and \( y \) are distinct elements of \( A \) and \( \mu_B(x) > 0 \), then \( E_A(x, y) < \mu_B(x) \).
3. \( \mu_B(c) = \mu_A(c) \), for any constant \( c \) in the language.

Then, a fuzzy relation \( E_B \) on \( \mu_B \) given by

\[
E_B(x, y) := E_A(x, y) \otimes \mu_B(x) \otimes \mu_B(y),
\]

is a compatible fuzzy equality on \( \mu_B \).
Let $\tilde{A} = (A, \mu_A, E_A, L)$ be a fuzzy algebra and $\mu_B : A \rightarrow L$ a fuzzy subalgebra of $A$, fulfilling the following conditions:

1. $\mu_B(x) \leq \mu_A(x)$ for all $x \in A$.
2. If $x$ and $y$ are distinct elements of $A$ and $\mu_B(x) > 0$, then $E_A(x, y) < \mu_B(x)$.
3. $\mu_B(c) = \mu_A(c)$, for any constant $c$ in the language.

Then, a fuzzy relation $E_B$ on $\mu_B$ given by

$$E_B(x, y) := E_A(x, y) \otimes \mu_B(x) \otimes \mu_B(y),$$

is a compatible fuzzy equality on $\mu_B$. 
Let $\bar{A} = (\mathcal{A}, \mu_A, E_A, L)$ be a fuzzy algebra and $\mu_B : A \rightarrow L$ a fuzzy subalgebra of $\mathcal{A}$, fulfilling the following:

1. $\mu_B(x) \leq \mu_A(x)$ for all $x \in A$.
2. If $x$ and $y$ are distinct elements from $A$ and if $\mu_B(x) > 0$, then $E_A(x, y) < \mu_B(x)$.
3. $\mu_B(c) = \mu_A(c)$, for any constant $c$.
4. $E_B(x, y) := E_A(x, y) \otimes \mu_B(x) \otimes \mu_B(y)$.

Then we say that the fuzzy algebra $\bar{B} = (\mathcal{A}, \mu_B, E_B, L)$ is a (fuzzy) subalgebra of the fuzzy algebra $\bar{A}$.
Let $\mathcal{M}$ be an equational class of fuzzy algebras and let $\bar{A} \in \mathcal{M}$ where $\bar{A} = (A, \mu_A, E_A, L)$. If $\bar{B} = (A, \mu_B, E_B, L)$ is a fuzzy subalgebra of $\bar{A}$, then also $\bar{B} \in \mathcal{M}$. 
Fuzzy identities and fuzzy equational classes

Fuzzy homomorphism

Operator H
Fuzzy identities and fuzzy equational classes

Fuzzy homomorphism

Operator H

Let $\bar{A} = (A, \mu_A, E_A, L)$ and $\bar{B} = (B, \mu_B, E_B, L)$ be fuzzy algebras of the same type. We say that $f : A \rightarrow B$ is a fuzzy mapping of $\bar{A}$ into $\bar{B}$ if the following conditions hold:

1. $(\forall a \in A) \mu_B(f(a)) \geq \mu_A(a)$
2. Let $t_1(x_1, ..., x_n)$, $t_2(x_1, ..., x_n)$ be terms in the language of $A$, let $t_1^A$, $t_2^A$ be the corresponding term operations and $a_1, ..., a_n$ elements from $A$.

If $E_A(t_1^A(a_1, ..., a_n), t_2^A(a_1, ..., a_n)) \geq \bigotimes_{i=1}^{n} \mu_A(a_i)$,

then $E_B(f(t_1^A(a_1, ..., a_n)), f(t_2^A(a_1, ..., a_n))) \geq \mu_B(f(t_1^A(a_1, ..., a_n))) \otimes \mu_B(f(t_2^A(a_1, ..., a_n)))$. 
Theorem

Let $\bar{A} = (A, \mu_A, E_A, L)$, $\bar{B} = (B, \mu_B, E_B, L)$ and $\bar{C} = (C, \mu_C, E_C, L)$ be fuzzy algebras of the same type and $f : A \rightarrow B$, $g : B \rightarrow C$ fuzzy mappings. Then also their composition $f \circ g : A \rightarrow C$ is a fuzzy mapping.
Let $\tilde{A} = (\mathcal{A}, \mu_A, E_A, L)$, $\tilde{B} = (\mathcal{B}, \mu_B, E_B, L)$ and $\tilde{C} = (\mathcal{C}, \mu_C, E_C, L)$ be fuzzy algebras of the same type and $f : A \to B$, $g : B \to C$ fuzzy mappings. Then also their composition $f \circ g : A \to C$ is a fuzzy mapping.

Let $\tilde{A} = (\mathcal{A}, \mu_A, E_A, L)$ and $\tilde{B} = (\mathcal{B}, \mu_B, E_B, L)$ be fuzzy algebras of the same type. Then a fuzzy mapping $h : A \to B$ is a fuzzy homomorphism of $\tilde{A}$ into $\tilde{B}$ if the following holds:

1. For each $n$-ary operation $f_A$ and for all $a_1, ..., a_n \in A$, $h(f_A(a_1, ..., a_n)) = f_B(h(a_1), ..., h(a_n))$.
2. $h(c_A) = c_B$, for every nullary operation $c$ in the language, $c_A$ and $c_B$ being the corresponding constants in $\mathcal{A}$ and $\mathcal{B}$ respectively.
Fuzzy identities and fuzzy equational classes

Fuzzy homomorphism

Proposition

Let $\bar{A} = (A, \mu_A, E_A, L)$ be a fuzzy algebra and $\bar{B} = (B, F_B)$ a crisp subalgebra of $A$. If

$$
\mu_B(x) := \begin{cases} 
\mu_A(x), & x \in B \\
0, & \text{else}
\end{cases} \quad \text{and}
$$

$$
E_B(x, y) := E_A(x, y) \otimes \mu_B(x) \otimes \mu_B(y),
$$

then $\bar{B} = (A, \mu_B, E_B, L)$ is a fuzzy subalgebra of $\bar{A}$. 
Theorem

Let \( \bar{A} = (A, \mu_A, E_A, L) \) and \( \bar{B} = (B, \mu_B, E_B, L) \) be fuzzy algebras and \( h : A \rightarrow B \) a fuzzy homomorphism. Define \( \bar{D} = (B, \mu_D, E_D, L) \), where

\[
\mu_D(d) := \begin{cases} \mu_B(d), & d \in D = h(A) \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
E_D(x, y) = E_B(x, y) \otimes \mu_D(x) \otimes \mu_D(y).
\]

Then, \( \bar{D} \) is a fuzzy subalgebra of the fuzzy algebra \( \bar{B} \).
Theorem

Let $\bar{A} = (A, \mu_A, E_A, L)$ and $\bar{B} = (B, \mu_B, E_B, L)$ be fuzzy algebras and $h : A \to B$ a fuzzy homomorphism. If $F(x_1, ..., x_n)$ is a term in the same language and $F^A, F^B$ the corresponding term operations in $A$ and $B$ respectively, then $h$ is a fuzzy homomorphism from the fuzzy algebra $(\bar{A}, F^A)$ into the fuzzy algebra $(\bar{B}, F^B)$. 
Fuzzy identities and fuzzy equational classes

Fuzzy homomorphism

Theorem

Let $\mathcal{M}$ be an equational class of fuzzy algebras. If $\bar{A} \in \mathcal{M}$ and $\bar{D}$ is a homomorphic image of $\bar{A}$, then also $\bar{D} \in \mathcal{M}$. 
Theorem

Let \( \bar{A}_i = (A_i, \mu_i, E_{A_i}) \) for all \( i \in I \) be a family of fuzzy algebras of the same type, \( A = \prod_{i \in I} A_i \) the direct product of algebras \( A_i \) and let the following holds for all \( g_1, g_2 \in \prod_{i \in I} A_i \):

If \( g_1 \neq g_2 \) and \( \bigotimes_{i \in I} \mu_i(g_1(i)) \neq 0 \), then \( \bigotimes_{i \in I} E_{A_i}(g_1(i), g_2(i)) \neq \bigotimes_{i \in I} \mu_i(g_1(i)) \).
Let \( \{ \overline{A}_i = (A_i, \mu_i, E_{A_i}) \mid i \in I \} \) be a family of fuzzy algebras of the same type, \( A = \prod_{i \in I} A_i \) the direct product of algebras \( A_i \) and let the following holds for all \( g_1, g_2 \in \prod_{i \in I} A_i \):

If \( g_1 \neq g_2 \) and \( \bigotimes_{i \in I} \mu_i(g_1(i)) \neq 0 \), then

\[
\bigotimes_{i \in I} E_{A_i}(g_1(i), g_2(i)) \neq \bigotimes_{i \in I} \mu_i(g_1(i)).
\]
Then the following holds: If

1. $\mu(g) := \bigotimes_{i \in I} \mu_i(g(i)), \ g \in \prod_{i \in I} A_i$ and
2. $E_A(g_1, g_2) := \bigotimes_{i \in I} E_{A_i}(g_1(i), g_2(i)); \ g_1, g_2 \in \prod_{i \in I} A_i,$

then $\tilde{A} = \prod_{i \in I} \tilde{A}_i := (A, \mu, E_A, L)$ is a fuzzy algebra.
Then the following holds: If
1. \( \mu(g) := \bigotimes_{i \in I} \mu_i(g(i)), \ g \in \prod_{i \in I} A_i \) and
2. \( E_A(g_1, g_2) := \bigotimes_{i \in I} E_{A_i}(g_1(i), g_2(i)); \ g_1, g_2 \in \prod_{i \in I} A_i, \)
then \( \bar{A} = \prod_{i \in I} \bar{A}_i := (A, \mu, E_A, L) \) is a fuzzy algebra.

The above fuzzy algebra \( \bar{A} := (A, \mu, E_A, L) \) is the \textbf{direct product} of fuzzy algebras \( \bar{A}_i, i \in I. \)
Theorem

If a fuzzy identity $E(u(x_1, \ldots, x_n), v(x_1, \ldots, x_n))$ holds in all fuzzy algebras $\bar{A}_i$, $i \in I$ of a fixed type, then also this fuzzy identity holds in their product $\bar{A} = \prod_{i \in I} \bar{A}_i$. 
**Theorem**

Let $\mathcal{M}$ be an equational class of fuzzy algebras. Then the following hold:

1. If $\bar{A} \in \mathcal{M}$, and $\bar{B}$ is a fuzzy subalgebra of $\bar{A}$, then $\bar{B} \in \mathcal{M}$.
2. If $\bar{A} \in \mathcal{M}$, and $\bar{D}$ is a homomorphic image of $\bar{A}$, then $\bar{D} \in \mathcal{M}$.
3. If for every $i \in I$, $f\bar{A}_i$ belongs to $\mathcal{M}$, then also $\prod_{i \in I} \bar{A}_i \in \mathcal{M}$. 
Thank you for your attention!