A note to stochastic Processes

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FSTA 2012, Liptovský Ján , January 30 - February 3, 2012

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Introduction

Motivation

Let

$$X_{t_1}, X_{t_2}, \ldots$$

be a sequence of random variables. This sequence is a stochastic process.

Let us suppose that t_i is time.

Our question is: How can we model this stochastic process?

There exist various approaches for modeling of such process

Introduction

We will compare two approaches:

- The domain of this process is a Boolean σ-algebra (the standard probability space) and then we use standard techniques for forecasting. It means that we use Kolmogorov's probability theory.
- The domain of this process is a quantum logic and then we use a calculus based on two dimensional states. It means, that we use quantum logics (quantum probability model) - compatible and non-compatible random events.

Introduction

- comparing the quality of predictions: quntum model versus classical model
- basic notions in a quantum logic - state, observable, expectation
- conditional states and s-maps (two-variable states), causality
- covariance, conditional expectation, sum of non-compatible observables

Quantum logic

Definition

Let $(L, 0_L, 1_L, \vee, \wedge, \perp)$ a σ-lattice with the greatest element 1_L and the smallest element 0_L. Let $\perp\colon L \to L$ be a unary operation on L with the following properties:

(a) for all $a \in L$ there is a unique $a^\perp \in L$ such that $(a^\perp)^\perp = a$ and $a \vee a^\perp = 1_L$;

(b) if $a, b \in L$ and $a \le b$ then $b^\perp \le a^\perp$;

(c) if $a, b \in L$ and $a \le b$ then $b = a \vee (a^\perp \wedge b)$ (orthomodular law).

Then $(L, 0_L, 1_L, \vee, \wedge, \perp)$ is said to be **an orthomodular σ-lattice**.

Quantum logic

1. **orthogonal** ($a \perp b$) if $a \leq b^{\perp}$,
2. **compatible** ($a \leftrightarrow b$) if $a = (a \wedge b) \vee (a \wedge b^{\perp})$ and $b = (a \wedge b) \vee (a^{\perp} \wedge b)$.

Definition

A map $m : L \to [0, 1]$ is called a σ**-additive state on** L, if $m(1_L) = 1$ and

$$m(\vee_{i \in I} a_i) = \sum_{i \in I} m(a_i)$$

for $a_i \in L$, $a_i \perp a_j$ $i \neq j$, $i \in I \subset N$

An orthomodular σ-lattice L is called a **quantum** σ **-logic** (briefly QL), if there exists a σ-additive state.

There exists orthomodular lattices with no state (R.Greechie).

Observables

Definition

Let L be a QL. A $\sigma-$homomorphism x from Borel sets to L ($\mathcal{B}(R)$), such that $x(R) = 1_L$ is called **an observable on** L.

Let us denote $\mathcal{O}$ the set of all observables on L.

Definition

Let L be a QL and x be an observable on L. Then

- $R(x) = \{x(E); E \in \mathcal{B}(R)\}$
  is called **the range of the observable** x on L;
- $\sigma(x) = \cap\{E \in \mathcal{B}(R); \quad x(E) = 1_L\}$
  is called **the spectrum of the observable** .

Discrete: if $\sigma(x)$ is an at most countable set (all discrete $\mathcal{O}_D$).
Finite: if $\sigma(x)$ is a finite set (all finite $\mathcal{O}_F$).

Let L be a QL. Let $x \in \mathcal{O}$ and m be a σ-additive state on L.

- A state respons to a probability measure.
- An observable respons to a random variable.
- $R(x)$ is a Boolean sub-σ-algebra of L
- $m_x(t) = m(x(-\infty, t))$, $t \in R$ respons to a probability distribution of x.
- $E_m(x)$ respons to the expectation of the observable x in the state m.

$$E_m(x) = \int_R t\, m(x(dt)),$$

if the integral exists. If $x \in \mathcal{O}_D$, then

$$E_m(x) = \sum_{t \in R} tm(x(\{t\})).$$

Definition

Let L be a QL, $L_0 \subset L - \{0_L\}$. Let $f : L \times L_c \to [0, 1]$ be a function fulfilling the following

1. for each $a \in L_0$ $f(.|a)$ is a σ-additive state on L;
2. for each $a \in L_0$ $f(a|a) = 1$;
3. for mutually orthogonal (at most countably many) elements $a_1, a_2, ..., \vee_i a_i \in L_0$ the following is satisfied

$$f(b| \vee_i a_i) = \sum_i f(b|a_i) f(a_i| \vee_i a_i).$$

Then f is called **a conditional state on** L.

Independence

Definition

Let L be a QL and let f be a conditional state on L. Let $a, c \in L_0 \subset L - \{0_L\}$ and let $b \in L$. We say that

b **is independent of** a **with respect to the state** $f(.|c)$

$(b \asymp_{f(.|c)} a)$ if $f(c|a) = 1$ and $f(b|c) = f(b|a)$.

Let $a, b, 1_L \in L_0$. Unlike the Kolmogorovian theory

$f(b|1_L) = f(b|a)$ does not imply $f(a|1_L) \neq f(a|b)$, in general.

Well-known Baye's Theorem may be violated in a quantum logic.

Two dimensional state - s-map

Definition

Let L be a QL. A map $p : L \times L \to [0, 1]$ will be called **an s-map on** L if the following conditions are fulfilled:

(s1) $p(1_L, 1_L) = 1$;

(s2) for all $a, b \in L$ if $a \perp b$ then $p(a, b) = 0$;

(s3) for all $a, b, c \in L$ if $a \perp b$ then

$$p(a \vee b, c) = p(a, c) + p(b, c) \quad p(c, a \vee b) = p(c, a) + p(c, b).$$

Let us denote: $\mathcal{P}$ the system of all s-maps on L, which are σ-additive in both variables.

$\mathcal{P}_S = \{p \in \mathcal{P}; p(a, b) = p(b, a) \quad \forall a, b \in L\}$, and $\mathcal{P}_N = \mathcal{P} - \mathcal{P}_S$.

Proposition. Let L be a QL and $p \in \mathcal{P}$. The following statements are true:

1. $\mu_p : L \to [0,1]$, such that $\mu_p(a) = p(a,a)$ is a σ-additive state on L.
2. For all $a, b \in L$ we have that $p(a,b) \leq p(a,a) = p(a,1_L)$.
3. If $a \leftrightarrow b$, then $p(a,b) = p(a \wedge b, 1_L)$.
4. For arbitrary $a, b \in L$ the following equivalence holds

$$f_p(b|1_L) = f_p(b|a) \quad \Leftrightarrow \quad p(b,a) = p(a,1_L)p(b,1_L).$$

5. Let $p \in \mathcal{P}$ and $L_0 = \{b \in L : \quad p(b,b) \neq 0\}$. Then

$$f_p(a,b) = \frac{p(a,b)}{p(b,b)}$$

is a conditional state $f_p : L \times L_0 \to [0,1]$.

Probability measure

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

$$P(A) = P(B) = 1 \text{ iff } P(A \cap B) = 1$$

State and s-map on a QL

Let L be a QL and m be a state and p be an s-map. Then

$$m(a) = m(b) = 1 \text{ does not imply } m(a \wedge b) = 1.$$

Jauch-Piron state: $m(a) = m(b) = 1$ iff $m(a \wedge b) = 1$.

$$p(a, a) = p(b, b) = 1 \text{ iff } p(a, b) = p(b, a) = 1.$$

and moreover $p(a, c) = p(c, a)$ for all $c \in L$.

Joint distribution

Definition

Let L be a QL and let $x, y \in \mathcal{O}$. Then a map $p_{x,y} : \mathcal{B}(R)^2 \to [0,1]$, such that $p_{x,y}(t,s) = p(x((-\infty,t)), y((-\infty,s)))$ is called a joint p-distribution for the observables x, y.

Definition

Let $x, y \in \mathcal{O}$. Let us denote

$$E_p(x,y) = \int\int_{R^2} t \cdot s \cdot p(x(dt), y(ds))$$

if the right-hand-side integral exists.
If $x, y \in \mathcal{O}_D$, then $E_p(x,y) = \sum_{t\in\sigma(x)} \sum_{s\in\sigma(y)} t \cdot s \cdot p(x(\{t\}), y(\{s\})$
whenever the right-hand-side sum exists.

Let us denote $c_p(x,y) = E_p(x,y) - E_p(x)E_p(y)$.

Proposition. Let L be a QL, $p \in \mathcal{P}$. For each $x, y \in \mathcal{O}_F$ there exist probability spaces $(\Omega_i, \mathcal{S}_i, P_i)$ (for $i = 1, 2$) and random variables ξ_i, η_i $\mathcal{S}_i$-measurable respectively, such that:

(r1) $E_i(\xi_i) = E_p(x)$ and $E_i(\eta_i) = E_p(y)$, $i = 1, 2$;

(r2) $c_p(x,y) = cov(\xi_1, \eta_1)$, $c_p(y,x) = cov(\eta_2, \xi_2)$;

(r3) $(c_p(x,y))^2 \le c_p(x,x)c_p(y,y)$.

Proposition. Let L be a QL, $p_1, p_2 \in \mathcal{P}$ and $p = \alpha p_1 + (1-\alpha)p_2$, $\alpha \in [0,1]$. If $p_1(a,a) = p_2(a,a)$ $\forall a \in L$, then $\forall x, y \in \mathcal{O}_F$

$$c_p(x,y) = \alpha c_{p_1}(x,y) + (1-\alpha)c_{p_2}(x,y);$$

Stochastic causality

Definition

Let L be a QL and $x, y \in \mathcal{O}$. Let $p \in \mathcal{P}$. We say that:

1. x **is causal to** y **with respect to** p if there exist some $A, B \in \mathcal{B}(R)$ such that
$$p(x(A), y(B)) \neq p(y(B), x(A));$$
2. x **is strong causal to** y **with respect to** p if for any $A, B \in \mathcal{B}(R)$
$$p(x(A), y(B)) = p(x(A), 1_L)p(y(B), 1_L)$$
and moreover there exist $A_0, B_0 \in \mathcal{B}(R)$ such that
$$p(y(B_0), x(A_0)) \neq p(y(B_0, 1_L))p(x(A_0, 1_L)).$$

Proposition. Let L be a QL and let $p_1, p_2 \in \mathcal{P}$, such that for $\forall u, v \in L$ $p_1(u,v) = p_2(v,u)$. If $x, y \in \mathcal{O}_D$ are strong causal, then there exists a probability space $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$, $\alpha \in [0,1]$ and random variables $\xi_{\alpha,t}$ such that

$$E_p(z(t)) = E(\xi_{\alpha,t}), \qquad E_p(z(t), z(t)) - E_p(z(t))^2 = D(\xi_{\alpha,t}),$$

where $t \in \{x, y\}$ and $z(x) = x$, $z(y) = y$. Moreover

$$cov(\xi_{\alpha,x}, \xi_{\alpha,y}) = \alpha c_p(x,y) + (1-\alpha) c_p(y,x) = c_{p,\alpha}(x,y).$$

The covariance matrix $\Sigma_{x,y}(\alpha)$ is positive semidefinite

$$\Sigma_{x,y}(\alpha) = \begin{pmatrix} c_p(x,x) & c_{p,\alpha}(x,y) \\ c_{p,\alpha}(y,x) & c_p(y,y) \end{pmatrix}.$$

We may transform the non-compatible observables x, y into one probability space. Their images are compatible. Thus we get the symmetric covariance matrix

$$\Sigma_{x,y}(0.5) = \frac{1}{2}.\left(\Sigma_{xy} + \Sigma_{xy}^T\right)$$

$$\Sigma_{x,y}(0.5) = \mathcal{A}(0.5) \circ (\Sigma_{xy} + \Sigma_{xy}^T) = \frac{1}{2}.\left(\Sigma_{xy} + \Sigma_{xy}^T\right)$$

Let x, y be strongly causal ($c_p(x, y) \neq 0$ and $c_p(y, x) = 0$). Then

$$\Sigma_{x,y}(0.5) = \begin{pmatrix} c_p(x,x) & \frac{1}{2}c_p(x,y) \\ \frac{1}{2}c_p(x,y) & c_p(y,y) \end{pmatrix}$$

Conditional expectation

Definition

Let L be a QL, $p \in \mathcal{P}$, $x \in \mathcal{O}$ and $\mathcal{B}$ be a Boolean sub-σ-algebra of L. **A version of conditional expectation of the observable** x **with respect to** $\mathcal{B}$ ($E_p(x|\mathcal{B}) = z$) is such an observable z that $R(z) \subset \mathcal{B}$ and $E_{f_p}(z|a) = E_{f_p}(x|a)$ for arbitrary

$$a \in \{u \in \mathcal{B}; \mu_p(u) \neq 0\}.$$

Since $R(x)$ is Boolean sub-σ-algebra of L we will write simply $E_p(y|x) = E_p(y|R(x))$.

Proposition. Let L be a QL, $p \in \mathcal{P}$ and $x, y \in \mathcal{O}_D$. Then the following statements are true:

1. $E_p(x, E_p(y|x)) = E_p(x, y)$.
2. $E_p(E_p(x|y)) = E_p(x)$;
3. $E_p(x|x) = x$;
4. $E_p(E_p(x|y)|y) = E_p(x|y)$;
5. $c_p(x, y) = c_p(E_p(x|y)), y)$.

“Sum” of observables in a quantum logic

Let L be a QL, $p \in \mathcal{P}$ and $x, y \in \mathcal{O}$.

Compatibility

If $x \leftrightarrow y$ then $x = f \circ h$ and $y = g \circ h$.
Loomis-Sikorski Theorem: $x + y = (f + g) \circ h$.

Non compatibility

If x, y are non-compatible then we cannot apply this procedure and $x + y$ does not exist in this sense.

Definition

Let L be a QL and $p \in \mathcal{P}$. A map $\oplus_p : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ is called *a summability operator* if the following conditions are fulfilled

(d1) $R(\oplus_p(x, y)) \subset R(y)$;

(d2) $\oplus_p(x, y) = E_p(x|y) + y$.

Definition

Let L be a QL, $\mathcal{B}$ be a Boolean sub-σ-algebra of L, and $p \in \mathcal{P}$. A map $\oplus_p^{\mathcal{B}} : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ is called *a summability operator with respect to a condition* $\mathcal{B}$ if the following conditions are fulfilled

(a1) $R\left(\oplus_p^{\mathcal{B}}(x, y)\right) \subset \mathcal{B}$;

(a2) $\oplus_p^{\mathcal{B}}(x, y) = E_p(x|\mathcal{B}) + E_p(y|\mathcal{B})$.

Proposition. Let L be a QL, $\mathcal{B}$ be a Boolean sub-σ-algebra of L, and $p \in \mathcal{P}$. Assume $x, y \in \mathcal{O}$. Then the following statements are satisfied

(e1) if $x \leftrightarrow y$ then $\oplus_p(x, y) \leftrightarrow \oplus_p(y, x)$;

(e2) $\oplus_p^{\mathcal{B}}(x, y) = \oplus_p^{\mathcal{B}}(y, x)$;

(e3) $E_p\left(\oplus_p^{\mathcal{B}}(x, y)\right) = E_p(\oplus_p(x, y)) = E_p(x) + E_p(y)$;

(e4) if $x, y \in \mathcal{O}_D$ then

$$E_p(x) + E_p(y) = \sum_{t\in\sigma(x)} \sum_{r\in\sigma(y)} (t + r)p(x(\{t\}), y(\{r\})).$$

Linear regression

Let $(\Omega, \mathcal{F}, P)$ a probability space and let ε be a random vector. Let

$$\mathbf{Y} = A\beta + \varepsilon,$$

where β be the vector of unknown parameters and

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & n \end{pmatrix}^T.$$

If residuals ε_i are autocorrelated or heteroscedastic, then we use generalized least squares (GLS) method. The GLS estimator of the coefficients in a linear model is

$$\hat{\beta} = (A^T \Sigma^{-1} A)^{-1} A^T \Sigma^{-1} Y.$$

The precise form of covariance matrix Σ depends on the nature of the errors process.
(Kubáček, L., Kubáčková, L., Volaufová J: Statistical models with linear structure. page 11, model (1.4))

Classical and quantum model

Two types of stochastic processes:
a Markov chain and **an AR-process**.
Comparison of predictions

1. the classical model - the covariance matrix Σ_1,
2. the quantum model - the covariance matrix $\Sigma_{\frac{1}{2}}$ from Σ_1: $cov(Z_i, Z_j) = 0$, $cov(Z_j, Z_i) = j$, $i > j$.

$$\Sigma_1 = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}, \qquad \Sigma_{\frac{1}{2}} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 & \dots & 1 \\ \frac{1}{2} & 1 & 3 & \dots & \frac{3}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & 1 & \frac{3}{2} & \dots & n \end{pmatrix}.$$

Classical and quantum model

Our statistical analysis was the following:

(SA1) We generated random variables $\varepsilon_i, \sim N(0,1)$ (i.i.d.), $(i = 1, ..., 15)$ which gave a Markov chain with the covariance matrix Σ_1 such that $Z_i = \sum_{j=1}^{i} \varepsilon_j$.

(SA2) From 10 values $(Z_1, \ldots, Z_{10})$ we computed prediction intervals for α%-confidence levels for 0.1 up to 0.99 with step 0.01, for each of the time instants $11, 12, \ldots, 15$. The prediction intervals were computed for for the classical model (Σ_1) and for the quantum model $(\Sigma_{\frac{1}{2}})$.

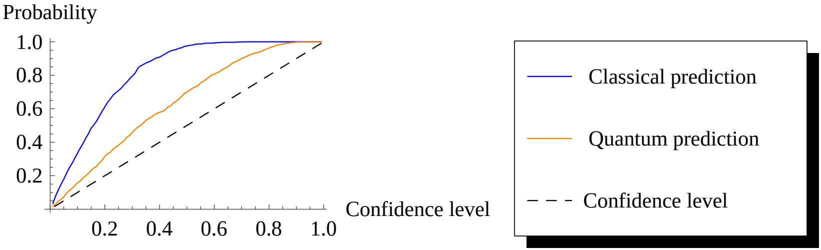
Classical and quantum model

(SA3) For each of the random variables $Z_{11}, \dots, Z_{15}$ we found the least α such that the corresponding Z_j was element of the α%-confidence interval.

(SA4) We repeated this procedure 1000 times and got the relative frequency for each of $Z_{11}, \dots, Z_{15}$ in prediction intervals for both models and each $0.01 \le \alpha \le 0.99$.

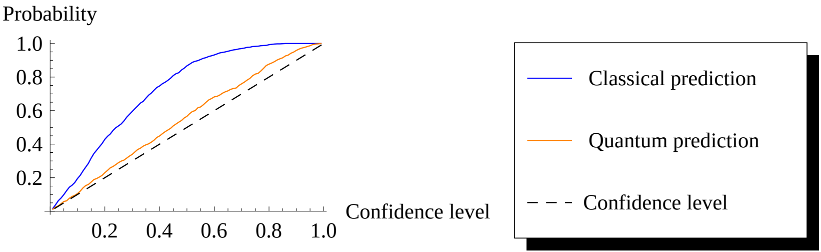
The results for Z_{11}, Z_{13}, Z_{15} are in the following figures.

Figure: Relative frequency (probability) of Z_{11} in prediction intervals for both models



We can see that for Z_{11}, the real confidence levels of prediction intervals for both models over-estimated, but confidence levels for the quantum model are less over-estimated (nearer to α's).

Figure: Relative frequency (probability) of Z_{13} in prediction intervals for both models



For Z_{13}, the real confidence levels of prediction intervals computed for the classical model are over-estimated, while the confidence levels for the quantum model are almost equal to α's.

AR process

an AR process with its correlation matrix Σ_{AR}

$$\Sigma_{AR} = \begin{pmatrix} 1 & \rho & \rho^2 & \vdots & \rho^{n-1} \\ \rho & 1 & \rho & \vdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \vdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \vdots & 1 \end{pmatrix}$$

Classical and quantum model

The correlation matrix for the quantum model of this AR process $\Sigma_{AR_{\frac{1}{2}}}$:
Σ_{AR} by by assumptions that for $i > j$ we have $cor(Y_i, Y_j) = 0$, and $cor(Y_j, Y_i) = \rho^{i-j}$.

$$\Sigma_{AR_{\frac{1}{2}}} = \begin{pmatrix} 1 & \frac{\rho}{2} & \frac{\rho^2}{2} & \dots & \frac{\rho^n}{2} \\ \frac{\rho}{2} & 1 & \frac{\rho}{2} & \dots & \frac{\rho^{n-1}}{2} \\ \frac{\rho^2}{2} & \frac{\rho}{2} & 1 & \dots & \frac{\rho^{n-2}}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\rho^n}{2} & \frac{\rho^{n-1}}{2} & \frac{\rho^{n-2}}{2} & \dots & 1 \end{pmatrix}$$

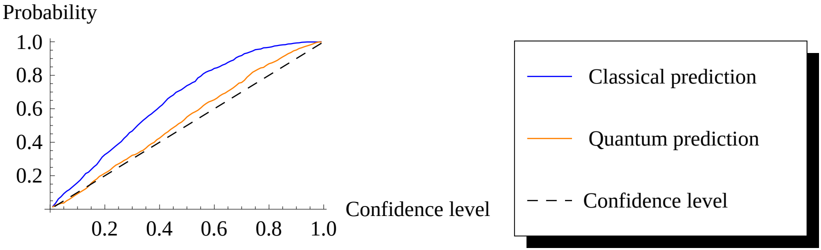
Classical and quantum model

Also in this case we will call the model based on the correlation matrix Σ_{AR} *classical*, the model based on the correlation matrix $\Sigma_{AR_{\frac{1}{2}}}$ *quantum*. We have generated the following process

$$Y_i = \rho Y_{i-1} + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma^2), \text{ I.I.D.}$$

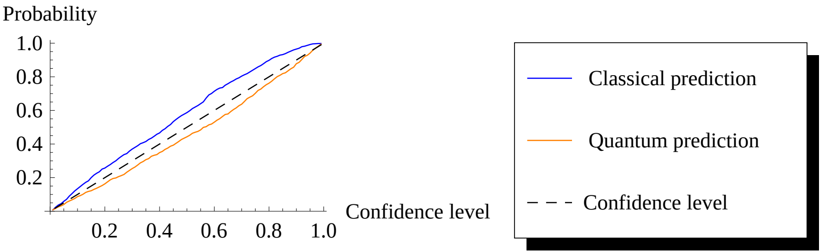
We made the statistical analysis for Y_t based on steps (SA1)-(SA4) as by the Markov chain. The results for Y_{11}, Y_{12}, Y_{13} are in the following figures. For details on prediction intervals for AR processes one can consult, e.g.

Figure: Relative frequency (probability) of Y_{11} in prediction intervals for both models



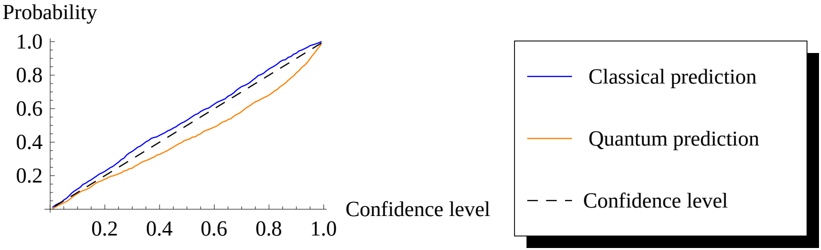
For Y_{11}, the real confidence levels of prediction intervals computed for the classical model are over-estimated, while the confidence levels for the quantum model are almost equal to α's.

Figure: Relative frequency (probability) of Y_{12} in prediction intervals for both models



For Y_{12}, the real confidence levels of prediction intervals computed both, for the classical as well as for the quantum models, are almost equal to α's, but the confidence levels for the classical model are 'a little' over-estimated, while the confidence levels for the quantum

Figure: Relative frequency (probability) of Y_{13} in prediction intervals for both models



For Y_{13}, the real confidence levels of prediction intervals computed for the classical model are almost equal to α's, while the confidence levels for the quantum model are under-estimated.

Thank you for your kind attention