Implications generated by increasing functions

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Motivation

Smutná, D.:
On many valued conjunctions and implications
Journal of Electrical Engineering (1999)

Jayaram B., Mesiar R.:
I-fuzzy equivalence relations and I-fuzzy partitions
Information Sciences (2009)

Implications generated using a function of one variable

- Yager's f-implications and g-implications (2004),
- Jayaram's h-generated implications (2006),
- Massanet's and Torrens' h- and generalized h-implications (2011),
- Massanet's and Torrens' (h, e)-implications and their generalization (2011),
- Massanet's and Torrens' e-generated and e-vertical generated implications,
- I_f, I_f^*-implications (1999, 2011).

Preliminaries

Fuzzy negation

A decreasing function $N : [0,1] \to [0,1]$ is called a negation if $N(0) = 1, N(1) = 0$. A fuzzy negation N is called

1. strict if it is strictly decreasing and continuous,
2. strong if it is an involution, i.e. $N(N(x)) = x$ for all $x \in [0,1]$.

A dual negation based on N is given by $N^d(x) = 1 - N(1-x)$.

Example

The standard negation $N_s(x) = 1 - x$ is strong. The negation $N(x) = 1 - x^2$ is strict, but not strong. The Gödel negation N_{G_1} is the least fuzzy negation and dual Gödel negation N_{G_2} is the greatest fuzzy negation, both are non-continuous negations:

$$N_{G_1} = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0, \end{cases} ; N_{G_2} = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Preliminaries

Fuzzy implication (Fodor and Roubens, 1994)

A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditons:

(I1) I is decreasing in its first variable,

(I2) I is increasing in its second variable,

(I3) $I(1,0) = 0$, $I(0,0) = I(1,1) = 1$.

Natural negations

Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implications. The function N_I defined by $N_I(x) = I(x,0)$ for all $x \in [0,1]$, is called the natural negation of I.

Preliminaries

A fuzzy implication $I : [0,1]^2 \to [0,1]$ satisfies:

(NP) the left neutrality property, or is called left neutral, if

$$I(1, y) = y; \quad y \in [0,1],$$

(EP) the exchange principle if

$$I(x, I(y,z)) = I(y, I(x,z)) \text{ for all } x, y, z \in [0,1],$$

(IP) the identity principle if

$$I(x, x) = 1; \quad x \in [0,1],$$

(OP) the ordering property if

$$x \leq y \iff I(x,y) = 1; \quad x, y \in [0,1],$$

(CP) the contrapositive symmetry with respect to a given negation N if

$$I(x,y) = I(N(y), N(x)); \quad x, y \in [0,1].$$

(LI) the law of importation with respect to a t-norm T if

$$I(T(x,y), z) = I(x, I(y,z)); \quad x, y, z \in [0,1].$$

Preliminaries

(S, N)-implication

A function $I : [0,1]^2 \to [0,1]$ is called an $(S, N)-$implication if there exist a t-conorm S and fuzzy negation N such that

$$I(x, y) = S(N(x), y), \quad x, y \in [0,1].$$

If N is a strong negation then I is called a strong implication.

Residual implication

with respect to a left-continuous triangular norm T

$$I_T(x, y) = \max\{z \in [0,1]; T(x, z) \leq y\}.$$

Preliminaries

Pseudo-Inverse

Let $g : [0,1] \to [0,\infty]$ be an increasing function. The function $g^{(-1)}$ which is defined by

$$g^{(-1)}(x) = \sup\{z \in \ [0,1]; g(z) < \ x\},$$

is called the pseudo-inverse of g with the convention $\sup\emptyset = 0$.

Generated Implications

Theorem (Smutná, 1999)

Let $g : [0,1] \to [0,\infty]$ *be a strictly increasing function such that* $g(0) = 0$. *Then the function* $I^g(x,y) : [0,1]^2 \to [0,1]$ *which is given by*

$$I^g(x,y) = g^{(-1)}(g(1-x) + g(y)),$$

is a fuzzy implication.

Theorem (Smutná, 1999)

Let $g : [0,1] \to [0,\infty]$ *be a strictly increasing function such that* $g(0) = 0$ *and* N *be a fuzzy negation. Then the function* $I_N^g : [0,1]^2 \to [0,1]$ *given as*

$$I_N^g(x,y) = g^{(-1)}(g(N(x)) + g(y)),$$

is a fuzzy implication.

Generated Implications

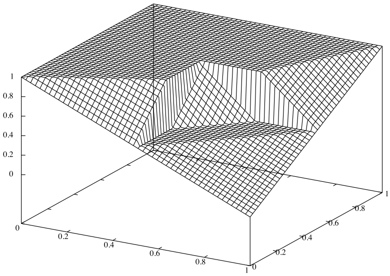
Let $g_1, g_2 : [0,1] \to [0,\infty]$ be given by

- $g_1(x) = \begin{cases} x & \text{if } x \leq 0.5, \\ 0.5 + 0.5x & \text{otherwise,} \end{cases}$
- $g_2(x) = -\ln(1-x)$.

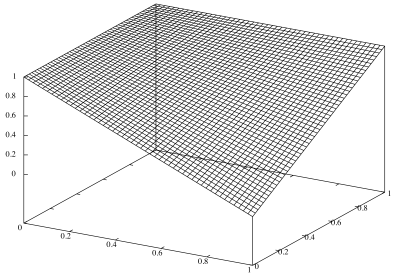
For functions $g_1^{(-1)}$ and $g_2^{(-1)}$ we get

- $g_1^{(-1)}(x) = \begin{cases} x & \text{if } x \leq 0,5, \\ 0,5 & \text{if } 0,5 < x \leq 0,75, \\ 2x-1 & \text{if } 0,75 < x \leq 1, \\ 1 & \text{if } 1 < x, \end{cases}$
- $g_2^{(-1)}(x) = 1 - e^{-x}$ for $x \in [0,\infty]$.

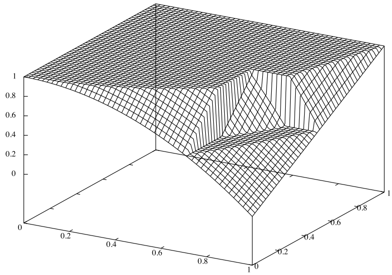
$$I^{g_1}(x,y) = \begin{cases} 1 - x + y & \text{if } x \geq 0.5, y \leq 0.5, x - y \geq 0.5, \\ 0.5 & \text{if } x \geq 0.5, y \leq 0.5, 0.25 \leq x - y < 0.5, \\ 1 - 2x + 2y & \text{if } x \geq 0.5, y \leq 0.5, x - y < 0.25, \\ \min(1 - x + 2y, 1) & \text{if } x < 0.5, y \leq 0.5, \\ \min(2 - 2x + y, 1) & \text{if } x \geq 0.5, y > 0.5, \\ 1 & \text{if } x < 0.5, y > 0.5, \end{cases}$$



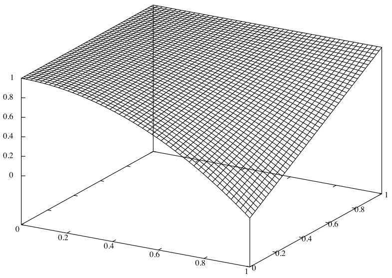
$$I^{g_2}(x,y) = 1 - e^{\ln(x(1-y))} = 1 - x + xy.$$



$$I_N^{g_1}(x,y) = \begin{cases} 1 - x^2 + y & \text{if } x \geq \frac{1}{\sqrt{2}}, y \leq 0.5, x^2 - y \geq 0.5, \\ 0.5 & \text{if } x \geq \frac{1}{\sqrt{2}}, y \leq 0.5, 0.25 \leq x^2 - y < 0.5, \\ 1 - 2x^2 + 2y & \text{if } x \geq \frac{1}{\sqrt{2}}, y \leq 0.5, x^2 - y < 0.25, \\ \min(1 - x^2 + 2y, 1) & \text{if } x < \frac{1}{\sqrt{2}}, y \leq 0.5, \\ \min(2 - 2x^2 + y, 1) & \text{if } x \geq \frac{1}{\sqrt{2}}, y > 0.5, \\ 1 & \text{if } x < \frac{1}{\sqrt{2}}, y > 0.5, \end{cases}$$



$$I_N^{g_2}(x, y) = 1 - x^2 + x^2 y.$$



Let c be a positive constant. If $g : [0, 1] \to [0, \infty]$ be a strictly increasing function with $g(0) = 0$ and N be an arbitrary negation. Then the implication I^g coincides with $I^{c \cdot g}$, and the implication I_N^g coincides with $I_N^{c \cdot g}$.

Lemma

Let $g : [0, 1] \to [0, \infty]$ be a strictly increasing function such that $g(0) = 0$. Then the natural negation related to I_g is $N_{I^g}(x) = 1 - x$.

Proposition

Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$. Then I^g satisfies (NP) and (CP) with respect to N_s.

(NP) the left neutrality property: $I(1,y) = y; \quad y \in [0,1]$,

(CP) the contrapositive symmetry with respect to a given negation N:

$$I(x,y) = I(N(y), N(x)); \quad x, y \in [0,1].$$

Proposition

Let $g : [0,1] \to [0,\infty]$ be a continuous and strictly increasing function such that $g(0) = 0$. Then I^g satisfies the (EP).

(EP) the exchange principle:

$$I(x, I(y,z)) = I(y, I(x,z)) \text{ for all } x, y, z \in [0,1],$$

There exist also non-continuous functions g such that I^g satisfy (EP).

Example

Let $g : [0,1] \to [0,1]$ be given by

$$g(x) = \begin{cases} 0 & x = 0, \\ \frac{1}{2}(x+1) & \text{otherwise.} \end{cases}$$

For its pseudo-inverse we get

$$g^{(-1)}(x) = \begin{cases} 0 & x \leq \frac{1}{2}, \\ 2x - 1 & x > \frac{1}{2}. \end{cases}$$

$$I^g(x,y) = \begin{cases} y & x = 1, \\ 1 - x & y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem

Let $g : [0,1] \to [0,\infty]$ *be a strictly increasing function continuous on* $]0,1]$ *such that* $g(0) = 0$*. Then* I^g *is an* (S,N)*-implication which is strong.*

I-fuzzy equivalence relation (Jayaram, Mesiar 2009)

Let I be an implication. A fuzzy subset E of the Cartesian product X^2 is called an I-fuzzy equivalence relation on X if the following properties are satisfied for all $x, y, z \in X$:

- E is reflexive; $E(x, x) = 1$,
- E is symmetric; $E(x, y) = E(y, x)$,
- E is $I-$transitive; $I(E(x, y), E(y, z)) \geq E(x, z)$.

For a given fuzzy implication I the concept of I-equivalence admits a value $a \in [0, 1]$ if there exist an I-equivalence $E : X^2 \to [0, 1]$ and elements $x, y \in X$ such that $E(x, y) = a$.

Proposition (Jayaram, Mesiar 2009)

For a given implication I the following are equivalent:

- The concept of $I-$equivalence relation admits any value $a \in [0,1]$.
- I satisfies the following
  $I(1,a) \geq a, \quad I(a,a) = 1.$

Proposition

Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0, g(1) = 1$. If function g is symmetric with respect to the point $\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)$, the concept of I^g-equivalence relation admits any value $a \in [0,1]$.

Proposition

Let $g : [0, 1] \to [0, \infty]$ be a strictly increasing function such that $g(0) = 0$ and $N : [0, 1] \to [0, 1]$ be a negation. If for each $x \in [0, 1]$ the following is true

$$g(x) + g(N(x)) \geq g(1),$$

then the concept of I_N^g-equivalence relation admits any value $a \in [0, 1]$.

Corollary

Let $g : [0, 1] \to [0, \infty]$ *be a strictly increasing function such that* $g(0) = 0, g(1) = 1$ *and* $N : [0, 1] \to [0, 1]$ *be a negation. If function* g *is symmetric with respect to the point* $\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)$ *and* $N(x) \geq 1 - x$ *for all* $x \in [0, 1]$*, the concept of* I_N^g-*equivalence relation admits any value* $a \in [0, 1]$.

Equivalence relation fitting implication (Jayaram, Mesiar 2009)

An implication I is called an equivalence relation fitting implication if each I-fuzzy equivalence relation E on X is consistent, i.e., if there exist $x, y \in X$ such that $E(x, y) = 1$, then for all $z \in X$

$$E(x, z) = E(y, z).$$

Proposition (Jayaram, Mesiar 2009)

If an implication is such that $I(1, a) = a$ and $I(a, a) = 1$ for any $a \in [0, 1]$, then I is an equivalence relation fitting implication.

Remark (Jayaram, Mesiar 2009)

If implication I satisfies (NP) and (IP), then I is an equivalence relation fitting implication.

Meaningful fuzzy partition (Jayaram, Mesiar 2009)

Each (fuzzy) equivalence relation $E : X^2 \to [0,1]$ induces a fuzzy partition $P_E = \{U_x; x \in X\}$ given by $U_x(y) = E(x,y)$. We say that the fuzzy partition P_E is *meaningful* if for all $x, y, z \in X$ we have that

$$E(x,y) = 1 \quad \Rightarrow \quad U_x = U_y.$$

Lemma (Jayaram, Mesiar 2009)

Let I be a fuzzy implication. Then I is an equivalence relation fitting fuzzy implication if and only if for all I-equivalence relations $E : X^2 \to [0,1]$ the induced fuzzy partition P_E is meaningful.

Example

Let us have $g(x) = x$. Then we get

$$I^g(x, y) = \min\{1, 1 - x + y\}.$$

We have $I^g(1, y) = y$ for all $y \in [0, 1]$. For an arbitrary non-void set X and an arbitrary I^g-equivalence relation $E_{I^g} : X^2 \to [0, 1]$ we get that if $E_{I^g}(a, b) = 1$ for $a, b \in X$ then

$$E_{I^g}(a, c) \leq I^g\left(1, E_{I^g}(b, c)\right) = E_{I^g}(b, c),$$

on the other hand

$$E_{I^g}(b, c) \leq I^g\left(1, E_{I^g}(a, c)\right) = E_{I^g}(a, c).$$

In this case the partition induced by E_{I^g} is meaningful.