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Plurivaluationistic Models of Vagueness in Logic-based Fuzzy Mathematics

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Fuzzy plurivaluationsm

NJJ Smith: *Vagueness and Degrees of Truth.* OUP 2008

- A good degree-theoretical semantic account of vagueness
- Some logical considerations missing

This talk: *Logic-based* fuzzy plurivaluationism

Theses:

- Membership functions of vague predicates are not uniquely determined.
- We need to take this fact into account in fuzzy models of vagueness (How?—By using formal fuzzy logic)

Formal semantics

The classical approach:

1. A set of models (model-theoretic structures for the language)

   Bivalent predicates . . . two-valued models
   (extensions = classical sharp sets)
   Gradual predicates . . . fuzzy models
   (extensions = membership functions)

2. A distinguished model representing the actual state of affairs
   (the actual world—the 'true' extensions of predicates)

Linguistic indeterminacy

Meaning-determining facts:

- Actual usage (application of predicates to things by speakers)
- Intentions of speakers
- Stipulative definitions, etc.

The tenet of fuzzy plurivaluationism:

In gradual predicates, the meaning-determining facts do not determine membership functions uniquely. Indeed:

There is *nothing* in language and its use that would determine whether a man of height 1.86 m is tall to degree 0.8 or 0.9

⇒ Instead of a single fuzzy model,
the meaning of a vague predicate is a *set of fuzzy models*

Plurivaluationistic formal semantics

For *tall*, the meaning-determining facts only determine that:

- Taller people have larger degrees of *tall*
- Certain people (e.g., Christopher Lee) are definitely *tall*
- Certain people (e.g., Michael J. Fox) are definitely *not tall*
- Small changes in height result in small changes of *tall*ness

$\Rightarrow$ Any monotone continue membership function (with certain boundary conditions) is admissible for *tall*

There is nothing in language or its use that would determine the meaning of *tall* more precisely

$\Rightarrow$ The meaning of *tall* $=$ a *set* of all admissible membership functions

Semantic indeterminacy

The degree of *John is tall* cannot be determined:
It varies across admissible models

John is tall has *no* unique truth degree:
There is no meaning-determining fact that would determine it

The semantics of vague predicates (such as *tall*) is

- Gradual (fuzzy) and
- Indeterminate (plurivaluationistic)

Slogan: Vagueness = graduality + indeterminacy

Fuzzy plurivaluationistic semantics of vagueness

Models based on single fuzzy sets:

- Address graduality, but neglect indeterminacy
- Only model *gradual precisifications* of vague predicates

Fuzzy plurivaluationism:

- Addresses both aspects of vagueness
- Solves the problem of artificial precision of fuzzy sets

  (precise degrees are *not* determined)

- Is theoretically sound, but there is a practical problem:

Degrees of vague properties (such as *tall*) cannot be determined

$\Rightarrow$ we cannot compute with them

Traditional fuzzy modeling

In fuzzy applications, particular membership functions are chosen

However, for most vague predicates this choice is arbitrary
(Recall: language does not determine unique membership functions, but only sets thereof)

$\Rightarrow$ Such models use *precisified technical meanings* of vague words

This may be efficient for applications, but the properties of the technical meanings may be just artifacts of the arbitrary choice

Living with plurivaluations

Q: Which of the properties of a technical precisification are not artifacts of the arbitrary choice of membership function, but do reflect the properties of the vague predicate?

A: Clearly only those that hold for *any* admissible choice of membership function!

Ie, those holding for the whole class of admissible models

Ie, just the *consequences* of the meaning-determining facts

Formal fuzzy logic is a tool tailored to derive these consequences

The role of formal fuzzy logic

Recall that the meaning-determining facts determine the following meaning postulates for *tall*:

- Taller people have larger degrees of *tall*
- Certain people (e.g., Christopher Lee) are definitely *tall*
- Certain people (e.g., Michael J. Fox) are definitely *not tall*
- Small changes in height result in small changes of *tall*ness

These meaning postulates can be formulated as a formal *theory* in fuzzy logic:

- $(h(x) \geq h(y)) \to (Ty \to Tx)$
- $Ta_1 \mathbin{\&} \neg Ta_0$
- $(h(x) \sim h(y)) \to (Tx \leftrightarrow Ty)$ (details omitted)

Admissible models are the models of this theory

Properties valid for all admissible models =
logical consequences of the theory (in formal fuzzy logic)

Adequate treatment of vagueness

$\Rightarrow$ Adequate degree-theoretic treatment of vague predicates =
deriving consequences in fuzzy logic, rather than
computing degrees in particular fuzzy models

Formal fuzzy logicians always implicitly did so:
modeling in formal fuzzy logic is done by axiomatic fuzzy theories
(and deriving theorems valid in all models)

The utility of formal fuzzy logic

Fuzzy logic is not indispensable for handling plurivaluations:
admissible models can as well be described by crisp conditions,
and ordinary mathematics used to derive their consequences
= the approach of traditional fuzzy mathematics

This approach is manageable with the technical precisifications,
but becomes too complicated for fuzzy plurivaluations

Example: fuzzy quantifiers

Many large mammals are critically endangered (Q P's are R's)

Traditional precisification: choose a membership function of

- *Large mammal* (a fuzzy set P)
- *Critically endangered* (a fuzzy set R)
- *Many* (a fuzzy relation Q between the fuzzy sets P, R)

. . . manageable in traditional fuzzy mathematics

Fuzzy plurivaluationistic model:

- *Large mammal* is a *set* of fuzzy sets
- *Critically endangered* is a *set* of fuzzy sets
- *Many* might be modeled as a

set of fuzzy relations between two sets of fuzzy sets

. . . hardly manageable in traditional fuzzy mathematics

. . . but well manageable in higher-order fuzzy logic

Conclusions

- Membership functions of vague predicates are not uniquely determined.

  We have to live with that.

- We need to take this fact into account in fuzzy models of vague predicates

  How?—By using formal fuzzy logic
  (instead of calculating particular degrees)