FSTA 2012
February 2, 2012
Liptovský Jan, the Slovak Republic

Distributivity of implications over t-representable operations in interval-valued fuzzy sets theory

Michał Baczyński

University of Silesia, Katowice, Poland

OUTLINE

- **Motivation and historical background**
- **Basic definitions and preliminary results**
- **Main equation for t-representable t-norms**
- **Some results pertaining to functional equations**
- **Conclusion and further work**

In the classical logic we have the following tautology:

$$(p \wedge q) \mapsto r \equiv (p \mapsto r) \vee (q \mapsto r).$$

In the classical logic we have the following tautology:

$$(p \wedge q) \mapsto r \equiv (p \mapsto r) \vee (q \mapsto r).$$

If we consider a generalization of this formula in (classical) fuzzy logic, then we obtain the following functional equation

$$I(T(x,y),z) = S(I(x,z), I(y,z)), \qquad x,y,z \in [0,1], \tag{1}$$

where

$T\colon [0,1]^2 \to [0,1]$ is some extension of classical conjunction (t-norm)

$S\colon [0,1]^2 \to [0,1]$ is some extension of classical disjunction (t-conorm)

$I\colon [0,1]^2 \to [0,1]$ is some extension of classical implication (fuzzy implication)

In the classical logic we have the following tautology:

$$(p \wedge q) \mapsto r \equiv (p \mapsto r) \vee (q \mapsto r).$$

If we consider a generalization of this formula in (classical) fuzzy logic, then we obtain the following functional equation

$$I(T(x,y),z) = S(I(x,z), I(y,z)), \qquad x,y,z \in [0,1], \tag{1}$$

where

$T\colon [0,1]^2 \to [0,1]$ is some extension of classical conjunction (t-norm)
$S\colon [0,1]^2 \to [0,1]$ is some extension of classical disjunction (t-conorm)
$I\colon [0,1]^2 \to [0,1]$ is some extension of classical implication (fuzzy implication)

W.E. Combs, J.E. Andrews (1998): Combinatorial rule explosion eliminated by a fuzzy rule configuration, *IEEE Trans. Fuzzy Systems* 6, 1–11 (1998)

They refer to the left-hand side of this equivalence as an intersection rule configuration (IRC) and to its right-hand side as a union rule configuration (URC).

S. Dick, A. Kandel (1999): Comments on "Combinatorial rule explosion eliminated by a fuzzy rule configuration", *IEEE Trans. Fuzzy Syst.* 7, 475–477:

> "Future work on this issue will require an examination of the properties of various combinations of fuzzy unions, intersections and implications."

S. Dick, A. Kandel (1999): Comments on "Combinatorial rule explosion eliminated by a fuzzy rule configuration", *IEEE Trans. Fuzzy Syst.* 7, 475–477:

> "Future work on this issue will require an examination of the properties of various combinations of fuzzy unions, intersections and implications."

J.M. Mendel, Q. Liang (1999): Comments on "Combinatorial rule explosion eliminated by a fuzzy rule configuration", *IEEE Trans. Fuzzy Syst.* 7, 369–371:

> "We think that what this all means is that we have to look past the mathematics of IRC⇔URC and inquire whether what we are doing when we replace IRC by URC makes sense."

Aczél (1966): general solutions of the distributive equation

$$F(x, G(y, z)) = G(F(x, z), F(y, z)),$$

when F is continuous and G is continuous, strictly increasing and associative.

Aczél (1966): general solutions of the distributive equation

$$F(x, G(y, z)) = G(F(x, z), F(y, z)),$$

when F is continuous and G is continuous, strictly increasing and associative.

E. Trillas & C. Alsina, (2002): On the Law $[p \wedge q \rightarrow r] = [(p \rightarrow r) \vee (q \rightarrow r)]$ in Fuzzy Logic, *IEEE Trans. Fuzzy Syst.* 10, 84–88.
Investigations on the equation (1):

$$I(T(x, y), z) = S(I(x, z), I(y, z)), \qquad x, y, z \in [0, 1],$$

in the case when T is a t-norm, S is a t-conorm and I is a fuzzy implication.
In the case of

- R-implications generated from left-continuous t-norms
- S-implications

the equation (1) holds if and only if $T = \min$ and $S = \max$.

We can consider other distributive laws for the classical implication:

$$
\begin{aligned}
(p \vee q) \mapsto r &\equiv (p \mapsto r) \wedge (q \mapsto r) \\
p \mapsto (q \wedge r) &\equiv (p \mapsto q) \wedge (p \mapsto r) \\
p \mapsto (q \vee r) &\equiv (p \mapsto q) \vee (p \mapsto r)
\end{aligned}
$$

We can consider other distributive laws for the classical implication:

$$(p \vee q) \mapsto r \equiv (p \mapsto r) \wedge (q \mapsto r)$$
$$p \mapsto (q \wedge r) \equiv (p \mapsto q) \wedge (p \mapsto r)$$
$$p \mapsto (q \vee r) \equiv (p \mapsto q) \vee (p \mapsto r)$$

All above equalities can be transformed to the functional equations of Pexider type:

$$I_1(T(x,y),z) = S(I_2(x,z), I_3(y,z)) \tag{D1}$$
$$I_1(S(x,y),z) = T(I_2(x,z), I_3(y,z)) \tag{D2}$$
$$I_1(x, T_1(y,z)) = T_2(I_2(x,y), I_3(x,z)) \tag{D3}$$
$$I_1(x, S_1(y,z)) = S_2(I_2(x,y), I_3(x,z)) \tag{D4}$$

Baczyński (2001, 2002): Eq. (D3) when $T_1 = T_2$ is a strict t-norm

Jayaram & Rao (2004): Eqs. (D2) – (D4) for R-implications and S-implications. In almost all the cases the distributivity holds only when $T_1 = T_2 = T = \min$ and $S_1 = S_2 = S = \max$

Ruiz-Aguilera & Torrens (2005, 2007): Distributivity of different classes of fuzzy implications over different classes of uninorms

Qin & Zhao (2005): Distributive equations for idempotent uninorms and nullnorms

Baczyński & Jayaram (2007, 2008, 2010): Distributivity of fuzzy implications over continuous Archimedean t-norms and t-conorms

Drewniak & Rak (2009): Subdistributivity and superdistributivity of binary op.

Baczyński (2010): Distributivity of fuzzy implications over representable uninorms

Qin & Yang (2010): Distributivity of fuzzy implications over nilpotent t-norms

Baczyński & Qin (2011): Distributivity of fuzzy implications over continuous t-norms

MAIN GOAL

The distributivity equations in:

- Atanassov's intuitionistic fuzzy sets theory
- in interval-valued fuzzy sets theory.

MAIN GOAL

The distributivity equations in:

- Atanassov's intuitionistic fuzzy sets theory
- in interval-valued fuzzy sets theory.

THIS CONTRIBUTION

Solutions of the following distributive equation

$$\mathcal{I}(x, \mathcal{T}_1(y, z)) = \mathcal{T}_2(\mathcal{I}(x, y), \mathcal{I}(x, z))$$

for t-representable t-norms in interval-valued fuzzy sets theory.

Definition 1 (Atanassov, 1999).

An (Atanassov's) intuitionistic fuzzy set A on X is a set

$$A = \{(x, \mu_A(x), \nu_A(x)) \ : \ x \in X\},$$

where $\mu_A, \ \nu_A \colon X \to [0,1]$ are called, respectively, the membership function and the non-membership function. Moreover, they satisfy the condition

$$\mu_A(x) + \nu_A(x) \leq 1, \qquad x \in X.$$

Definition 1 (Atanassov, 1999).

An (Atanassov's) intuitionistic fuzzy set A on X is a set

$$A = \{(x, \mu_A(x), \nu_A(x)) \quad : \quad x \in X\},$$

where $\mu_A,\ \nu_A \colon X \to [0,1]$ are called, respectively, the membership function and the non-membership function. Moreover, they satisfy the condition

$$\mu_A(x) + \nu_A(x) \leq 1, \qquad\qquad x \in X.$$

An (Atanassov's) intuitionistic fuzzy set A on X can be represented by the $\mathcal{L}^*$-fuzzy set A in the sense of Goguen given by

$$\begin{aligned} A \colon X &\to L^* \\ x &\mapsto (\mu_A(x), \nu_A(x)), \qquad\qquad x \in X, \end{aligned}$$

where $\mathcal{L}^* = (L^*, \leq_{L^*})$ is the following complete lattice

$$L^* = \{(x_1, x_2) \in [0,1]^2 \quad : \quad x_1 + x_2 \leq 1\}$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \geq y_2$$

with the units $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.

Another extension of the fuzzy sets theory is interval-valued fuzzy sets theory introduced, independently, by **Sambuc (1975) & Gorzałczany (1987)**.
We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where

$$L^I = \{(x_1, x_2) \in [0,1]^2 : x_1 \leq x_2\}$$
$$(x_1, x_2) \leq_{L^I} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1 \wedge x_2 \leq y_2$$

Another extension of the fuzzy sets theory is interval-valued fuzzy sets theory introduced, independently, by **Sambuc (1975) & Gorzałczany (1987)**.
We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where

$$L^I = \{(x_1, x_2) \in [0,1]^2 : x_1 \leq x_2\}$$
$$(x_1, x_2) \leq_{L^I} (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \leq y_2$$

In the sequel, if $x \in L^I$, then we denote it by $x = [x_1, x_2]$.
$\mathcal{L}^I$ is a complete lattice with units $0_{\mathcal{L}^I} = [0, 0]$ and $1_{\mathcal{L}^I} = [1, 1]$.

Definition 2. An interval-valued fuzzy set on X is a mapping $A \colon X \to L^I$.

An interval-valued fuzzy set can be seen as a $\mathcal{L}^I$-fuzzy set in the sense of Goguen.

Another extension of the fuzzy sets theory is interval-valued fuzzy sets theory introduced, independently, by **Sambuc (1975) & Gorzałczany (1987)**.
We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where

$$L^I = \{(x_1, x_2) \in [0,1]^2 : x_1 \leq x_2\}$$
$$(x_1, x_2) \leq_{L^I} (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \leq y_2$$

In the sequel, if $x \in L^I$, then we denote it by $x = [x_1, x_2]$.
$\mathcal{L}^I$ is a complete lattice with units $0_{\mathcal{L}^I} = [0,0]$ and $1_{\mathcal{L}^I} = [1,1]$.

Definition 2. An interval-valued fuzzy set on X is a mapping $A\colon X \to L^I$.

An interval-valued fuzzy set can be seen as a $\mathcal{L}^I$-fuzzy set in the sense of Goguen.

Deschrijver & Kerre (2003): Atanassov's intuitionistic fuzzy sets can be seen as interval-valued fuzzy sets (and vice-versa).

In our talk we develop our investigations in the terms of $\mathcal{L}^I = (L^I, \leq_{L^I})$, since the main results are easier to obtain and to show.

Definition 3. Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. An associative, commutative operation $\mathcal{T} \colon L^2 \to L$ is called a t-norm on $\mathcal{L}$ if it is increasing and $1_{\mathcal{L}}$ is the neutral element of $\mathcal{T}$.

Definition 3. Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. An associative, commutative operation $\mathcal{T}\colon L^2 \to L$ is called a t-norm on $\mathcal{L}$ if it is increasing and $1_{\mathcal{L}}$ is the neutral element of $\mathcal{T}$.

T-norms on $\mathcal{L}^I$

T-norms on $\mathcal{L}^I$ can be defined in many ways. In our talk we consider only the following special class of t-norms.

Definition 4.

A t-norm $\mathcal{T}$ on $\mathcal{L}^I$ is called t-representable if there exist t-norms T_1 and T_2 on $([0,1], \leq)$ such that $T_1 \leq T_2$ and

$$\mathcal{T}([x_1, x_2], [y_1, y_2]) = [T_1(x_1, y_1), T_2(x_2, y_2)], \qquad [x_1, x_2], [y_1, y_2] \in L^I.$$

Definition 3. Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. An associative, commutative operation $\mathcal{T}: L^2 \to L$ is called a t-norm on $\mathcal{L}$ if it is increasing and $1_{\mathcal{L}}$ is the neutral element of $\mathcal{T}$.

T-norms on $\mathcal{L}^I$

T-norms on $\mathcal{L}^I$ can be defined in many ways. In our talk we consider only the following special class of t-norms.

Definition 4.

A t-norm $\mathcal{T}$ on $\mathcal{L}^I$ is called t-representable if there exist t-norms T_1 and T_2 on $([0,1], \leq)$ such that $T_1 \leq T_2$ and

$$\mathcal{T}([x_1, x_2], [y_1, y_2]) = [T_1(x_1, y_1), T_2(x_2, y_2)], \qquad [x_1, x_2], [y_1, y_2] \in L^I.$$

It should be noted that not all t-norms on $\mathcal{L}^I$ are t-representable.

Definition 5 (Kitainik (1993); Fodor & Roubens (1994)).
Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. A function $\mathcal{I}\colon L^2 \to L$ is called a fuzzy implication on $\mathcal{L}$ if

- it is decreasing with respect to the first variable,
- it is increasing with respect to the second variable
- it fulfills the following conditions:

$$\mathcal{I}(0_{\mathcal{L}}, 0_{\mathcal{L}}) = \mathcal{I}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}, \qquad \mathcal{I}(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}.$$

Definition 5 (Kitainik (1993); Fodor & Roubens (1994)).
Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. A function $\mathcal{I}\colon L^2 \to L$ is called a fuzzy implication on $\mathcal{L}$ if

- it is decreasing with respect to the first variable,
- it is increasing with respect to the second variable
- it fulfills the following conditions:

$$\mathcal{I}(0_{\mathcal{L}}, 0_{\mathcal{L}}) = \mathcal{I}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}, \qquad \mathcal{I}(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}.$$

Directly from the above definition we can deduce that each implication $\mathcal{I}$ on $\mathcal{L}$ satisfies also the normality condition $\mathcal{I}(0_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$. Consequently, every implication restricted to the set $\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}^2$ coincides with the classical implication.

When $\mathcal{L} = ([0,1], \leq)$, then $\mathcal{I}$ is called a fuzzy implication,

When $\mathcal{L} = \mathcal{L}^I$, then $\mathcal{I}$ is called an interval-valued fuzzy implication.

Main equation

$$\boxed{\mathcal{I}(x, \mathcal{T}_1(y, z)) = \mathcal{T}_2(\mathcal{I}(x, y), \mathcal{I}(x, z))} \tag{2}$$

when t-norms $\mathcal{T}_1 = (T_1, T_2)$ and $\mathcal{T}_2 = (T_3, T_4)$ on $\mathcal{L}^I$ are t-representable.

Main equation

$$\boxed{\mathcal{I}(x, \mathcal{T}_1(y, z)) = \mathcal{T}_2(\mathcal{I}(x, y), \mathcal{I}(x, z))} \qquad (2)$$

when t-norms $\mathcal{T}_1 = (T_1, T_2)$ and $\mathcal{T}_2 = (T_3, T_4)$ on $\mathcal{L}^I$ are t-representable.

Assume that projection mappings on $\mathcal{L}^I$ are defined as the following:

$$pr_1([x_1, x_2]) = x_1, \qquad pr_2([x_1, x_2]) = x_2, \qquad \text{for } [x_1, x_2] \in L^I.$$

Main equation

$$\boxed{\mathcal{I}(x, \mathcal{T}_1(y, z)) = \mathcal{T}_2(\mathcal{I}(x, y), \mathcal{I}(x, z))} \tag{2}$$

when t-norms $\mathcal{T}_1 = (T_1, T_2)$ and $\mathcal{T}_2 = (T_3, T_4)$ on $\mathcal{L}^I$ are t-representable.

Assume that projection mappings on $\mathcal{L}^I$ are defined as the following:

$$pr_1([x_1, x_2]) = x_1, \qquad pr_2([x_1, x_2]) = x_2, \qquad \text{for } [x_1, x_2] \in L^I.$$

At this situation our distributive equation has the following form

$$\begin{aligned} \mathcal{I}([x_1, x_2],&[T_1(y_1, z_1), T_2(y_2, z_2)]) \\ &= [T_3(pr_1(\mathcal{I}([x_1, x_2], [y_1, y_2])), pr_1(\mathcal{I}([x_1, x_2], [z_1, z_2]))), \\ &\qquad T_4(pr_2(\mathcal{I}([x_1, x_2], [y_1, y_2])), pr_2(\mathcal{I}([x_1, x_2], [z_1, z_2])))], \end{aligned}$$

for all $[x_1, x_2], [y_1, y_2], [z_1, z_2] \in L^I$.

$$
\begin{aligned}
&pr_1(\mathcal{I}([x_1, x_2],[T_1(y_1, z_1), T_2(y_2, z_2)])) \\
&\qquad\qquad = T_3(pr_1(\mathcal{I}([x_1, x_2], [y_1, y_2])), pr_1(\mathcal{I}([x_1, x_2], [z_1, z_2]))), \\
&pr_2(\mathcal{I}([x_1, x_2],[T_1(y_1, z_1), T_2(y_2, z_2)])) \\
&\qquad\qquad = T_4(pr_2(\mathcal{I}([x_1, x_2], [y_1, y_2])), pr_2(\mathcal{I}([x_1, x_2], [z_1, z_2])))
\end{aligned}
$$

$$
\begin{aligned}
&pr_1(\mathcal{I}([x_1, x_2],[T_1(y_1, z_1), T_2(y_2, z_2)])) \\
&\qquad\qquad = T_3(pr_1(\mathcal{I}([x_1, x_2], [y_1, y_2])), pr_1(\mathcal{I}([x_1, x_2], [z_1, z_2]))), \\
&pr_2(\mathcal{I}([x_1, x_2],[T_1(y_1, z_1), T_2(y_2, z_2)])) \\
&\qquad\qquad = T_4(pr_2(\mathcal{I}([x_1, x_2], [y_1, y_2])), pr_2(\mathcal{I}([x_1, x_2], [z_1, z_2])))
\end{aligned}
$$

Now, let us fix arbitrarily $[x_1, x_2] \in L^I$ and define two functions $L^I \to L^I$ by

$$
g^1_{[x_1,x_2]}(\cdot) := pr_1 \circ \mathcal{I}([x_1, x_2], \cdot), \qquad g^2_{[x_1,x_2]}(\cdot) := pr_2 \circ \mathcal{I}([x_1, x_2], \cdot).
$$

$$
\begin{aligned}
&pr_1(\mathcal{I}([x_1, x_2],[T_1(y_1, z_1), T_2(y_2, z_2)])) \\
&\qquad = T_3(pr_1(\mathcal{I}([x_1, x_2], [y_1, y_2])), pr_1(\mathcal{I}([x_1, x_2], [z_1, z_2]))), \\
&pr_2(\mathcal{I}([x_1, x_2],[T_1(y_1, z_1), T_2(y_2, z_2)])) \\
&\qquad = T_4(pr_2(\mathcal{I}([x_1, x_2], [y_1, y_2])), pr_2(\mathcal{I}([x_1, x_2], [z_1, z_2])))
\end{aligned}
$$

Now, let us fix arbitrarily $[x_1, x_2] \in L^I$ and define two functions $L^I \to L^I$ by

$$
g^1_{[x_1,x_2]}(\cdot) := pr_1 \circ \mathcal{I}([x_1, x_2], \cdot), \qquad g^2_{[x_1,x_2]}(\cdot) := pr_2 \circ \mathcal{I}([x_1, x_2], \cdot).
$$

Then we get the following two equations

$$
\begin{aligned}
g^1_{[x_1,x_2]}([T_1(y_1, z_1), T_2(y_2, z_2)]) &= T_3(g^1_{[x_1,x_2]}([y_1, y_2]), g^1_{[x_1,x_2]}([z_1, z_2])), \\
g^2_{[x_1,x_2]}([T_1(y_1, z_1), T_2(y_2, z_2)]) &= T_4(g^2_{[x_1,x_2]}([y_1, y_2]), g^2_{[x_1,x_2]}([z_1, z_2])).
\end{aligned}
$$

$$pr_1(\mathcal{I}([x_1,x_2],[T_1(y_1,z_1),T_2(y_2,z_2)])) \\ = T_3(pr_1(\mathcal{I}([x_1,x_2],[y_1,y_2])), pr_1(\mathcal{I}([x_1,x_2],[z_1,z_2]))),$$

$$pr_2(\mathcal{I}([x_1,x_2],[T_1(y_1,z_1),T_2(y_2,z_2)])) \\ = T_4(pr_2(\mathcal{I}([x_1,x_2],[y_1,y_2])), pr_2(\mathcal{I}([x_1,x_2],[z_1,z_2])))$$

Now, let us fix arbitrarily $[x_1,x_2] \in L^I$ and define two functions $L^I \to L^I$ by

$$g^1_{[x_1,x_2]}(\cdot) := pr_1 \circ \mathcal{I}([x_1,x_2],\cdot), \qquad g^2_{[x_1,x_2]}(\cdot) := pr_2 \circ \mathcal{I}([x_1,x_2],\cdot).$$

Then we get the following two equations

$$g^1_{[x_1,x_2]}([T_1(y_1,z_1),T_2(y_2,z_2)]) = T_3(g^1_{[x_1,x_2]}([y_1,y_2]), g^1_{[x_1,x_2]}([z_1,z_2])),$$
$$g^2_{[x_1,x_2]}([T_1(y_1,z_1),T_2(y_2,z_2)]) = T_4(g^2_{[x_1,x_2]}([y_1,y_2]), g^2_{[x_1,x_2]}([z_1,z_2])).$$

When $T_1 = T_2 = T_3 = T_4$, then in both cases we have the bisymmetry equation.
The continuous and strictly increasing solutions are known even for domain L^I (**Kocsis (2007)**: *A bisymmetry equation on restricted domain*).
But in our investigation t-norms are not strictly increasing on the whole domain.

We have solved (almost completely) this problem for the case when $T_1 = T_2$ and $T_3 = T_4$ are continuous and Archimedean t-norms on unit interval.

We have solved (almost completely) this problem for the case when $T_1 = T_2$ and $T_3 = T_4$ are continuous and Archimedean t-norms on unit interval.

Theorem 1. *A function $T\colon [0,1]^2 \to [0,1]$ is a nilpotent t-norm if and only if there exists a continuous, strictly decreasing function $t\colon [0,1] \to [0,\infty)$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0))), \qquad x, y \in [0,1].$$

We can transform our problem to the following equation (we deal only with g^1):

$$
\begin{aligned}
& g^1_{[x_1,x_2]}([t_1^{-1}(\min(t_1(y_1)+t_1(z_1),t_1(0))),t_1^{-1}(\min(t_1(y_2)+t_1(z_2),t_1(0)))]) \\
&= t_3^{-1}(\min(t_3(g^1_{[x_1,x_2]}([y_1,y_2]))+t_3(g^1_{[x_1,x_2]}([z_1,z_2])),t_3(0))).
\end{aligned}
$$

We can transform our problem to the following equation (we deal only with g^1):

$$g^1_{[x_1,x_2]}([t_1^{-1}(\min(t_1(y_1)+t_1(z_1),t_1(0))),t_1^{-1}(\min(t_1(y_2)+t_1(z_2),t_1(0)))]) \\ = t_3^{-1}(\min(t_3(g^1_{[x_1,x_2]}([y_1,y_2]))+t_3(g^1_{[x_1,x_2]}([z_1,z_2])),t_3(0))).$$

Hence

$$t_3 \circ g^1_{[x_1,x_2]}([t_1^{-1}(\min(t_1(y_1)+t_1(z_1),t_1(0))),t_1^{-1}(\min(t_1(y_2)+t_1(z_2),t_1(0)))]) \\ = \min(t_3 \circ g^1_{[x_1,x_2]}([y_1,y_2])+t_3 \circ g^1_{[x_1,x_2]}([z_1,z_2]),t_3(0)).$$

We can transform our problem to the following equation (we deal only with g^1):

$$g^1_{[x_1,x_2]}([t_1^{-1}(\min(t_1(y_1)+t_1(z_1),t_1(0))),t_1^{-1}(\min(t_1(y_2)+t_1(z_2),t_1(0)))]) \\ = t_3^{-1}(\min(t_3(g^1_{[x_1,x_2]}([y_1,y_2]))+t_3(g^1_{[x_1,x_2]}([z_1,z_2])),t_3(0))).$$

Hence

$$t_3\circ g^1_{[x_1,x_2]}([t_1^{-1}(\min(t_1(y_1)+t_1(z_1),t_1(0))),t_1^{-1}(\min(t_1(y_2)+t_1(z_2),t_1(0)))]) \\ = \min(t_3\circ g^1_{[x_1,x_2]}([y_1,y_2])+t_3\circ g^1_{[x_1,x_2]}([z_1,z_2]),t_3(0)).$$

Let $L^a=\{(u_1,u_2)\in[0,a]^2\ :\ u_1\geq u_2\}$, for real $a>0$.

Let us put $t_1(y_1)=u_1$, $t_1(y_2)=u_2$, $t_1(z_1)=v_1$ and $t_1(z_2)=v_2$.

$$[y_1,y_2],[z_1,z_2]\in L^I\Longrightarrow(u_1,u_2),(v_1,v_2)\in L^{t_1(0)}$$

If we put

$$f_{[x_1,x_2]}(u,v) := t_3 \circ pr_1 \circ \mathcal{I}([x_1,x_2],[t_1^{-1}(u),t_1^{-1}(v)]),$$

where $u,v \in [0,t_1(0)],\ u \geq v$, then we get the following functional equation

$$\begin{aligned} f_{[x_1,x_2]}(\min(u_1+v_1,t_1(0)),\min(u_2+v_2,t_1(0))) \\ = \min(f_{[x_1,x_2]}(u_1,u_2)+f_{[x_1,x_2]}(v_1,v_2),t_3(0)), \end{aligned}$$

satisfied for all $(u_1,u_2),(v_1,v_2) \in L^{t_1(0)}$. Of course function $f_{[x_1,x_2]}\colon L^{t_1(0)} \to [0,t_3(0)]$ is unknown above.

If we put

$$f_{[x_1,x_2]}(u,v) := t_3 \circ pr_1 \circ \mathcal{I}([x_1,x_2],[t_1^{-1}(u),t_1^{-1}(v)]),$$

where $u, v \in [0, t_1(0)],\ u \geq v$, then we get the following functional equation

$$\begin{aligned} f_{[x_1,x_2]}&(\min(u_1+v_1,t_1(0)),\min(u_2+v_2,t_1(0))) \\ &= \min(f_{[x_1,x_2]}(u_1,u_2)+f_{[x_1,x_2]}(v_1,v_2),t_3(0)), \end{aligned}$$

satisfied for all $(u_1,u_2),(v_1,v_2) \in L^{t_1(0)}$. Of course function $f_{[x_1,x_2]} \colon L^{t_1(0)} \to [0, t_3(0)]$ is unknown above.

In a same way we can repeat all the above calculations, but for the function g^2, to obtain the following functional equation

$$\begin{aligned} f^{[x_1,x_2]}&(\min(u_1+v_1,t_1(0)),\min(u_2+v_2,t_1(0))) \\ &= \min(f^{[x_1,x_2]}(u_1,u_2)+f^{[x_1,x_2]}(v_1,v_2),t_3(0)), \end{aligned}$$

satisfied for all $(u_1,u_2),(v_1,v_2) \in L^{t_1(0)}$, where

$$f^{[x_1,x_2]}(u,v) := t_3 \circ pr_2 \circ \mathcal{I}([x_1,x_2],[t_1^{-1}(u),t_1^{-1}(v)])$$

is an unknown function.

Results pertaining to functional equations

$$L^\infty = \{(u_1, u_2) \in [0, \infty]^2 \mid u_1 \geq u_2\}$$

$$L^a = \{(u_1, u_2) \in [0, a]^2 \mid u_1 \geq u_2\}$$

Results pertaining to functional equations

$L^\infty = \{(u_1, u_2) \in [0, \infty]^2 \mid u_1 \geq u_2\}$

$L^a = \{(u_1, u_2) \in [0, a]^2 \mid u_1 \geq u_2\}$

Solutions when $T_1 = T_2$ and $T_3 = T_4$ are strict t-norms

$$f(u_1 + v_1, u_2 + v_2) = f(u_1, u_2) + f(v_1, v_2) \tag{3}$$

$f\colon L^\infty \to [0, \infty]$ is an unknown function.

Results pertaining to functional equations

$L^\infty = \{(u_1, u_2) \in [0, \infty]^2 \mid u_1 \geq u_2\}$

$L^a = \{(u_1, u_2) \in [0, a]^2 \mid u_1 \geq u_2\}$

Solutions when $T_1 = T_2$ and $T_3 = T_4$ are strict t-norms

$$f(u_1 + v_1, u_2 + v_2) = f(u_1, u_2) + f(v_1, v_2) \tag{3}$$

$f\colon L^\infty \to [0, \infty]$ is an unknown function.

Solutions when $T_1 = T_2$ and $T_3 = T_4$ are nilpotent t-norms

$$h(\min(u_1 + v_1, a), \min(u_2 + v_2, a)) = \min(h(u_1, u_2) + h(v_1, v_2), b) \tag{4}$$

$h\colon L^a \to [0, b]$ is an unknown function.

Solutions when $T_1 = T_2$ is a nilpotent t-norm and $T_3 = T_4$ is a strict t-norm

$$g(\min(u_1 + v_1, a), \min(u_2 + v_2, a)) = g(u_1, u_2) + g(v_1, v_2) \tag{5}$$

$g\colon L^a \to [0, \infty]$ is an unknown function.

Solutions when $T_1 = T_2$ is a nilpotent t-norm and $T_3 = T_4$ is a strict t-norm

$$g(\min(u_1 + v_1, a), \min(u_2 + v_2, a)) = g(u_1, u_2) + g(v_1, v_2) \tag{5}$$

$g\colon L^a \to [0, \infty]$ is an unknown function.

Solutions when $T_1 = T_2$ is a strict t-norm and $T_3 = T_4$ is a nilptent t-norm

$$k(u_1 + v_1, u_2 + v_2) = \min(k(u_1, u_2) + k(v_1, v_2), b) \tag{6}$$

$k\colon L^\infty \to [0, b]$ is an unknown function.

Proposition 1 (Baczyński, Jayaram (2009)).
Fix real $a, b > 0$. *For a function* $f\colon [0, a] \to [0, b]$ *the following statements are equivalent:*

(i) f *satisfies the functional equation*

$$f(\min(x + y, a)) = \min(f(x) + f(y), b), \qquad x, y \in [0, a].$$

(ii) Either $f = 0$, *or* $f = b$, *or* $f(x) = \begin{cases} 0, & \text{if } x = 0, \\ b, & \text{if } x > 0, \end{cases}$, *or there exists a unique constant* $c \in \left[\frac{b}{a}, \infty\right)$ *such that* $f(x) = \min(cx, b)$, *for all* $x \in [0, a]$.

Main results

$$\boxed{\mathcal{I}(x, \mathcal{T}_1(y, z)) = \mathcal{T}_2(\mathcal{I}(x, y), \mathcal{I}(x, z))}$$

Using this method we are able to solve our main equation when t-norms $\mathcal{T}_1 = (T_1, T_2)$ and $\mathcal{T}_2 = (T_3, T_4)$ on $\mathcal{L}^I$ are t-representable and such that $T_1 = T_2$ and $T_3 = T_4$ are continuous and Archmidean t-norms.

Main results

$$\boxed{\mathcal{I}(x, \mathcal{T}_1(y, z)) = \mathcal{T}_2(\mathcal{I}(x, y), \mathcal{I}(x, z))}$$

Using this method we are able to solve our main equation when t-norms $\mathcal{T}_1 = (T_1, T_2)$ and $\mathcal{T}_2 = (T_3, T_4)$ on $\mathcal{L}^I$ are t-representable and such that $T_1 = T_2$ and $T_3 = T_4$ are continuous and Archmidean t-norms.

FURTHER WORK IN THIS TOPIC

- Detailed description of all correct solutions
- Other distributive equations on $\mathcal{L}^I$ for t-representable operations
- Other classes of t-norms
- Possible applications

Thank You for the attention!