Diferenciálna geometria plochy

Užívateľské funkcie

derivácia vektorovej funkcie:

$$D_u(f,u,v,ord) := \begin{array}{|l} n \leftarrow \text{length}(f(u,v)) \\ \text{for } i \in 1..n \\ \quad vec_i \leftarrow \frac{d^{ord}}{du^{ord}}(f(u,v))_i \\ vec \end{array} \qquad D_v(f,u,v,ord) := \begin{array}{|l} n \leftarrow \text{length}(f(u,v)) \\ \text{for } i \in 1..n \\ \quad vec_i \leftarrow \frac{d^{ord}}{dv^{ord}}(f(u,v))_i \\ vec \end{array}$$

gradient funkcie troch premennych:

$$\text{grad}(f,x,y,z,ord) := \left(\frac{d^{ord}}{dx^{ord}}f(x,y,z) \quad \frac{d^{ord}}{dy^{ord}}f(x,y,z) \quad \frac{d^{ord}}{dz^{ord}}f(x,y,z)\right)^T$$

Vyšetrovaný bod: $(u_0 \ \ v_0) := (0 \ \ 0)$

elipsoid s polosami a,b,c > 0

Vektorová rovnica plochy a jej derivácie:

$$Pl(u,v) := (a\cdot\cos(u)\cdot\cos(v) \quad b\cdot\cos(u)\sin(v) \quad c\cdot\sin(u))^T \qquad Pl(u_0,v_0)^T \rightarrow (a \ \ 0 \ \ 0)$$

$$Pl_u(u,v) := D_u(Pl,u,v,1) \qquad Pl_u(u,v)^T \rightarrow (-a\cdot\sin(u)\cdot\cos(v) \quad -b\cdot\sin(u)\cdot\sin(v) \quad c\cdot\cos(u))$$

$$Pl_u(u_0,v_0)^T \rightarrow (0 \ \ 0 \ \ c)$$

$$Pl_v(u,v) := D_v(Pl,u,v,1) \qquad Pl_v(u,v)^T \rightarrow (-a\cdot\cos(u)\cdot\sin(v) \quad b\cdot\cos(u)\cdot\cos(v) \quad 0)$$

$$Pl_v(u_0,v_0)^T \rightarrow (0 \ \ b \ \ 0)$$

$$Pl_{uu}(u,v) := D_u(Pl,u,v,2) \qquad Pl_{uu}(u,v)^T \rightarrow (-a\cdot\cos(u)\cdot\cos(v) \quad -b\cdot\cos(u)\cdot\sin(v) \quad -c\cdot\sin(u))$$

$$Pl_{uu}(u_0,v_0)^T \rightarrow (-a \ \ 0 \ \ 0)$$

$$Pl_{vv}(u,v) := D_v(Pl,u,v,2) \qquad Pl_{vv}(u,v)^T \rightarrow (-a\cdot\cos(u)\cdot\cos(v) \quad -b\cdot\cos(u)\cdot\sin(v) \quad 0)$$

$$Pl_{vv}(u_0,v_0)^T \rightarrow (-a \ \ 0 \ \ 0)$$

$$Pl_{uv}(u,v) := D_v(Pl_u,u,v,1) \qquad Pl_{uv}(u,v)^T \rightarrow (a\cdot\sin(u)\cdot\sin(v) \quad -b\cdot\sin(u)\cdot\cos(v) \quad 0)$$

$$Pl_{uv}(u_0,v_0)^T \rightarrow (0 \ \ 0 \ \ 0)$$

Vektor normály a dotyková rovina:

$$nvek_\sigma(u,v) := \frac{Pl_u(u,v)\times Pl_v(u,v)}{|Pl_u(u,v)\times Pl_v(u,v)|} \qquad nvek_\sigma(u_0,v_0)^T \rightarrow \left(-c\cdot\frac{b}{\text{abs}(c\cdot b)} \quad 0 \quad 0\right) \qquad nvek_\sigma(u_0,v_0)^T \text{ assume}, a>0, b>0, c>0 \rightarrow (-1 \ \ 0 \ \ 0)$$

$$norm_\sigma(u,v,\lambda) := Pl(u,v) + \lambda\cdot nvek_\sigma(u,v) \qquad norm_\sigma(u_0,v_0,\lambda)^T \rightarrow \left(a - c\cdot\frac{b}{\text{abs}(c\cdot b)}\cdot\lambda \quad 0 \quad 0\right) \qquad norm_\sigma(u_0,v_0,\lambda)^T \text{ assume}, a>0, b>0, c>0 \rightarrow (a-\lambda \quad 0 \quad 0)$$

$$drov_\sigma(u,v,x,y,z) := \left((x \ \ y \ \ z)^T - Pl(u,v)\right)\cdot nvek_\sigma(u,v) \qquad drov_\sigma(u_0,v_0,x,y,z) \rightarrow -(x-a)\cdot c\cdot\frac{b}{\text{abs}(c\cdot b)} \qquad drov_\sigma(u_0,v_0,x,y,z) \text{ assume}, a>0, b>0, c>0 \rightarrow -x + a$$

Implicitné vyjadrenie plochy (značené "I" pred názvom)

parametre a vyšetrovaný bod:

$$Ia := (a \ \ b \ \ c \ \ 1)^T \qquad (x_0 \ \ y_0 \ \ z_0) := Pl(u_0,v_0)^T \qquad (x_0 \ \ y_0 \ \ z_0) \rightarrow (a \ \ 0 \ \ 0)$$

Poznámka.
Keďže $IPl = 0$, v našom špecifickom prípade (elipsoid) musí

$$Ia_4 = \frac{x^2}{(Ia_1)^2} + \frac{y^2}{(Ia_2)^2} + \frac{z^2}{(Ia_3)^2}.$$

rovnica plochy: (IPl = 0)

$$IPl(x,y,z) := \frac{x^2}{(Ia_1)^2} + \frac{y^2}{(Ia_2)^2} + \frac{z^2}{(Ia_3)^2} - Ia_4 \qquad IPl(x_0,y_0,z_0) \rightarrow 0$$

gradient (vektor normály):

$$\text{grad}(IPl,x,y,z,1)^T \rightarrow \left(2\cdot\frac{x}{a^2} \quad 2\cdot\frac{y}{b^2} \quad 2\cdot\frac{z}{c^2}\right)$$

$$Invek_\sigma(x,y,z) := \frac{\text{grad}(IPl,x,y,z,1)}{|\text{grad}(IPl,x,y,z,1)|} \qquad \text{grad}(IPl,x_0,y_0,z_0,1)^T \rightarrow \left(\frac{2}{a} \quad 0 \quad 0\right)$$

normála plochy:

$$Inorm_\sigma(x,y,z,\lambda) := IPl(x,y,z) + \lambda\cdot Invek_\sigma(x,y,z) \qquad Inorm_\sigma(x_0,y_0,z_0,I\lambda)^T \text{ assume}, a>0, b>0, c>0 \rightarrow (I\lambda \ \ 0 \ \ 0)$$

Pozn.: $I\lambda = a - \lambda$ (porovn. $norm_\sigma$)

dotyková rovina: ($Idrov_\sigma = 0$)

$$Idrov_\sigma(x0,y0,z0,x,y,z) := \left((x \ \ y \ \ z)^T - (x0 \ \ y0 \ \ z0)^T\right)\cdot Invek_\sigma(x0,y0,z0)$$

$$Idrov_\sigma(x_0,y_0,z_0,x,y,z) \text{ assume}, a>0, b>0, c>0 \rightarrow x - a$$

Pomocné členy pre ďalší výpočet

$$E(u,v) := Pl_u(u,v)\cdot Pl_u(u,v) \qquad E(u_0,v_0) \rightarrow c^2$$

$$F(u,v) := Pl_u(u,v)\cdot Pl_v(u,v) \qquad F(u_0,v_0) \rightarrow 0$$

$$G(u,v) := Pl_v(u,v)\cdot Pl_v(u,v) \qquad G(u_0,v_0) \rightarrow b^2$$

$$L(u,v) := nvek_\sigma(u,v)\cdot Pl_{uu}(u,v) \qquad L(u_0,v_0) \rightarrow c\cdot\frac{b}{\text{abs}(c\cdot b)}\cdot a$$

$$M(u,v) := nvek_\sigma(u,v)\cdot Pl_{uv}(u,v) \qquad M(u_0,v_0) \rightarrow 0$$

$$N(u,v) := nvek_\sigma(u,v)\cdot Pl_{vv}(u,v) \qquad N(u_0,v_0) \rightarrow c\cdot\frac{b}{\text{abs}(c\cdot b)}\cdot a$$

$$H(u,v) := \frac{E(u,v)\cdot N(u,v) + L(u,v)\cdot G(u,v) - 2\cdot F(u,v)\cdot M(u,v)}{2\left(E(u,v)\cdot G(u,v) - F(u,v)^2\right)}$$

$$H(u_0,v_0) \text{ assume}, a>0, b>0, c>0 \rightarrow \frac{1}{2}\cdot\frac{c^2\cdot a + b^2\cdot a}{c^2\cdot b^2}$$

$$K(u,v) := \frac{L(u,v)\cdot N(u,v) - M(u,v)^2}{E(u,v)\cdot G(u,v) - F(u,v)^2} \qquad K(u_0,v_0) \rightarrow \frac{1}{\text{abs}(c\cdot b)^2}\cdot a^2$$

Základné formy plochy

$$ZF_\sigma(u,v,du,dv) := \begin{pmatrix} E(u,v)\,du^2 + 2\,F(u,v)\,du\cdot dv + G(u,v)\,dv^2 \\ L(u,v)\cdot du^2 + 2\cdot M(u,v)\cdot du\cdot dv + N(u,v)\cdot dv^2 \end{pmatrix}$$

$$ZF_\sigma(u_0,v_0,du,dv) \text{ assume}, a>0, b>0, c>0 \rightarrow \begin{pmatrix} c^2\cdot du^2 + b^2\cdot dv^2 \\ a\cdot du^2 + a\cdot dv^2 \end{pmatrix}$$

Krivosti

normálová:

$$nkriv_\sigma(u,v,du,dv) := \frac{ZF_\sigma(u_0,v_0,du,dv)_2}{ZF_\sigma(u_0,v_0,du,dv)_1}$$

$$nkriv_\sigma(u_0,v_0,du,dv) \left| \begin{matrix}\text{assume}, a>0, b>0, c>0 \\ \text{simplify}\end{matrix} \right. \rightarrow a\cdot\frac{du^2+dv^2}{c^2\cdot du^2 + b^2\cdot dv^2}$$

hlavné:

$$hkriv_\sigma(u,v) := H(u,v) + \begin{pmatrix}1\\-1\end{pmatrix}\cdot\sqrt{H(u,v)^2 - K(u,v)}$$

$$hkriv_\sigma(u_0,v_0) \left| \begin{matrix}\text{assume}, a>0, b>0, c>0 \\ \text{simplify}\end{matrix} \right. \rightarrow \begin{pmatrix} \frac{1}{2}\cdot a\cdot\frac{\text{signum}(c^2-b^2)\cdot c^2 - \text{signum}(c^2-b^2)\cdot b^2 + c^2 + b^2}{c^2\cdot b^2} \\ \frac{-1}{2}\cdot a\cdot\frac{\text{signum}(c^2-b^2)\cdot c^2 - \text{signum}(c^2-b^2)\cdot b^2 - c^2 - b^2}{c^2\cdot b^2} \end{pmatrix}$$

úplná:

$$ukriv_\sigma(u,v) := hkriv_\sigma(u,v)_1\cdot hkriv_\sigma(u,v)_2$$

$$ukriv_\sigma(u_0,v_0) \left| \begin{matrix}\text{assume}, a>0, b>0, c>0 \\ \text{simplify}\end{matrix} \right. \rightarrow \frac{-1}{4}\cdot a^2\cdot\left(-\text{signum}(-c^2+b^2)\cdot c^2 + \text{signum}(-c^2+b^2)\cdot b^2 + c^2 + b^2\right)\cdot\frac{-\text{signu}}{}$$

stredná:

$$skriv_\sigma(u,v) := \frac{1}{2}\left(hkriv_\sigma(u,v)_1 + hkriv_\sigma(u,v)_2\right)$$

$$skriv_\sigma(u_0,v_0) \text{ assume}, a>0, b>0, c>0 \rightarrow \frac{1}{2}\cdot\frac{c^2\cdot a + b^2\cdot a}{c^2\cdot b^2}$$

Hlavné smery

rovnica:
($rovhs_\sigma = 0$)

$$rovhs_\sigma(u,v,u',v') := \begin{pmatrix} L(u,v)\cdot F(u,v) - M(u,v)\cdot E(u,v) \\ L(u,v)\cdot G(u,v) - N(u,v)\cdot E(u,v) \\ M(u,v)\cdot G(u,v) - N(u,v)\cdot F(u,v) \end{pmatrix}^T \cdot \begin{pmatrix} u'^2 \\ u'\cdot v' \\ v'^2 \end{pmatrix}$$

$$rovhs_\sigma(u_0,v_0,u',v') \text{ assume}, a>0, b>0, c>0 \rightarrow (b^2\cdot a - c^2\cdot a)\cdot u'\cdot v'$$

podmienka prirodzenej parametrizacie:
($ppp_\sigma = 1$)

$$ppp_\sigma(u,v,u',v') := \left| \begin{matrix} v \leftarrow Pl_u(u,v)\cdot u' + Pl_v(u,v)\cdot v' \\ \sqrt{v\cdot v} \end{matrix} \right.$$

$$ppp_\sigma(u_0,v_0,u',v') \rightarrow \left(b^2\cdot v'^2 + c^2\cdot u'^2\right)^{\frac{1}{2}}$$

riešenie sústavy 2rovníc o 2neznámych:

Given $\quad rovhs_\sigma(u,v,u',v') = 0$

$\quad ppp_\sigma(u,v,u',v') = 1$

$$u'v'_\sigma(u,v,u',v') := \text{Find}(u',v')$$

$$u'v'_\sigma(u_0,v_0,u',v') \rightarrow \begin{pmatrix} 0 & 0 & \frac{-1}{c} & \frac{1}{c} \\ \frac{1}{b} & \frac{-1}{b} & 0 & 0 \end{pmatrix}$$

hlavné smery:

$$hsmery_\sigma(u,v,u',v') := \left| \begin{matrix} korene \leftarrow u'v'_\sigma(u,v,u',v')^T \\ Pl_u(u,v)\cdot korene^{\langle 1\rangle T} + Pl_v(u,v)\cdot korene^{\langle 2\rangle T} \end{matrix} \right.$$

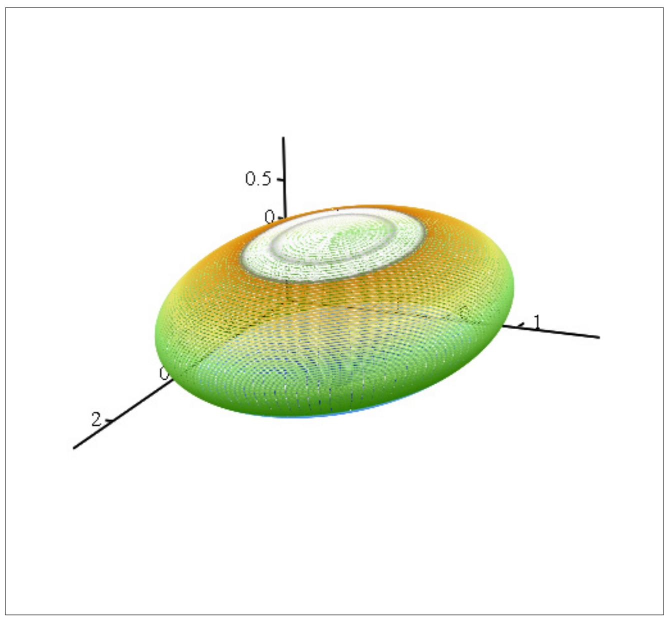
$$hsmery_\sigma(u_0,v_0,u',v') \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Pozn.:
Hlavné smery sú ktorékoľvek 2 vzájomne lineárne nezávislé stĺpcové vektory z matice "$hsmery_\sigma$", teda napríklad:

s1=(0,0,1)
s2=(0,1,0)

Grafické znázornenie: $\quad a := 3 \quad b := 2 \quad c := 1$

$$Plot(u,v) := (a\cdot\cos(u)\cdot\cos(v) \quad b\cdot\cos(u)\sin(v) \quad c\cdot\sin(u))^T$$



Plot